A sufficient condition for the existence of plane spanning trees on geometric graphs

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Let \( P \) be a set of \( n \geq 3 \) points in general position in the plane and let \( G \) be a geometric graph with vertex set \( P \). If the number of empty triangles \( \Delta uvw \) in \( P \) for which the subgraph of \( G \) induced by \( \{u, v, w\} \) is not connected is at most \( n - 3 \), then \( G \) contains a non-self intersecting spanning tree.

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1. Introduction

Throughout this article \( P \) denotes a set of \( n \geq 3 \) points in general position in the Euclidean plane. A geometric graph with vertex set \( P \) is a graph \( G \) drawn in such a way that each edge is a straight line segment with both ends in \( P \). A plane spanning tree of \( G \) is a non-self intersecting subtree of \( G \) that contains every vertex of \( G \). Plane spanning trees with or without specific conditions have been studied by various authors.

A well known result of Károlyi et al. [3] asserts that if the edges of a finite complete geometric graph \( G_{Kn} \) are coloured by two colours, then there exists a plane spanning tree of \( G_{Kn} \) all of whose edges are of the same colour. Keller et al. [4] characterized those plane spanning trees \( T \) of \( G_{Kn} \) such that the complement graph \( T^c \) contains no plane spanning trees.

A plane spanning tree \( T \) is a geometric independency tree if for each pair \( \{u, v\} \) of leaves of \( T \), there is an edge \( xy \) of \( T \) such that the segments \( uv \) and \( xy \) cross each other. Kaneko et al. [2] proved that every complete geometric graph with \( n \geq 5 \) vertices contains a geometric independence tree with at least \( n - 5 \) leaves.

Let \( k \) be an integer with \( 2 \leq k \leq 5 \) and \( G \) be a geometric graph with \( n \geq k \) vertices such that all geometric subgraphs of \( G \) induced by \( k \) vertices have a plane spanning tree. Rivera-Campo [6] proved that \( G \) has a plane spanning tree.

Three points \( u, v \) and \( w \) in \( P \) form an empty triangle if no point of \( P \) lies in the interior of the triangle \( \Delta uvw \). For any geometric graph \( G \) with vertex set \( P \) we say that an empty triangle \( \Delta uvw \) of \( P \) is disconnected in \( G \) if the subgraph of \( G \) induced by \( \{u, v, w\} \) is not connected.

Let \( s(G) \) denote the number of disconnected empty triangles of \( G \). Our result is the following:

**Theorem 1.** If \( G \) is a geometric graph with \( n \geq 3 \) vertices such that \( s(G) \leq n - 3 \), then \( G \) has a plane spanning tree.
For each $n \geq 3$, let $u_1, u_2, \ldots, u_n$ be the vertices of a regular $n$-gon and denote by $T_n$ and $T_n^c$ the plane path $u_1, u_2, \ldots, u_n$ and its complement, respectively. The geometric graph $T_n^c$ contains no plane spanning tree and is such that $s(T_n^c) = n - 2$. This shows that the condition in Theorem 1 is tight.

2. Proof of Theorem 1

For every oriented straight line $L$ we denote by $L^-$ the set of points in $P$ which are on or to the left of $L$ and by $L^+$ the points which are on or to the right of $L$.

A $k$-set of $P$ is a subset $X$ of $P$ with $k$ elements that can be obtained by intersecting $P$ with an open half plane. The main tool in the proof of Theorem 1 is the following procedure of Erdős et al. [5,1], used to generate all $k$-sets of $P$: Let $L = L_1$ be an oriented line passing through precisely one point $v_1$ of $P$ with $|L^-_1| = k + 1$. Rotate $L$ clockwise around the axis $v_1$ by an angle $\theta$ until a point $v_2$ in $P$ is reached. Now rotate $L$ in the same direction but around $v_2$ until a point $v_3$ in $P$ is reached, and continue rotating $L$ in a similar fashion obtaining a set of oriented lines $C(L)$ and a sequence of points $v_1, v_2, \ldots, v_s$, not necessarily distinct, where $v_s = v_1$ when the angle of rotation $\theta$ reaches $2\pi$.

For $i = 1, 2, \ldots, s - 1$, let $L(v_i, v_{i+1})$ be the line in $C(L)$ that passes through points $v_i$ and $v_{i+1}$ and for $i = 2, 3, \ldots, s - 1$, let $L_i$ be any line in $C(L)$ between $L(v_{i-1}, v_i)$ and $L(v_i, v_{i+1})$.

It is well known that for each line $L_i$ either $L_{i+1}^- = L_i^-$ and $L_{i+1}^+ = (L_i^- \setminus \{v_j\}) \cup \{v_{j+1}\}$, or $L_{i+1}^+ = (L_i^+ \setminus \{v_j\}) \cup \{v_{j+1}\}$ and $L_{i+1}^- = L_i^+$. In both cases $|L_{i+1}^-| = |L_i^-| = k + 1$ and $|L_{i+1}^+| = |L_i^+| = n - k$. It is also easy to see that if $v_{j+1} \in L_i^+$, then $L_i^-(v_j, v_{j+1}) = L_i^- \cup \{v_{j+1}\}$ and $L_i^+(v_j, v_{j+1}) = L_i^+$, and if $v_{j+1} \in L_i^-$, then $L_i^-(v_j, v_{j+1}) = L_i^-$ and $L_i^+(v_j, v_{j+1}) = (L_i^- \setminus \{v_j\}) \cup \{v_{j+1}\}$.

The following lemma will used in the proof of Theorem 1.

Lemma 2. Let $L_i, L_j \in C(L)$ with $i < j$. If $x, y$ and $z$ are points of $P$ lying in $L_i^+ \cap L_j^-$, then there are integers $k$ and $l$ with $i < k < l < j$ such that $v_k \in \{x, y, z\}$, $x, y, z \in L_k^+ \cap L_l^-$ and such that $L_i$ crosses the triangle $\Delta xyz$.

Proof. Consider the lines $L_i, L_{i+1}, \ldots, L_j$. The result follows from the fact that at each step $t$, at most one of the points $x, y, z$ switches from $L_t^+$ to $L_{t+1}^-$, see Fig. 1. □

Let $G$ be a geometric graph with $n \geq 3$ vertices such that $s(G) \leq n - 3$ and let $P$ denote the vertex set of $G$. If $n = 3$ or $n = 4$, it is not difficult to verify by inspection that $G$ has a plane spanning tree. Let us proceed with the proof of Theorem 1 by induction and assume $n \geq 5$ and that the result is valid for each geometric subgraph of $G$ with $k$ vertices, where $3 \leq k \leq n - 1$.

Let $v_1$ be a point in $P$ and $L_1$ be an oriented line through $v_1$ such that $|L_1^-| = \lceil \frac{n+1}{2} \rceil$ and $|L_1^+| = \lfloor \frac{n+1}{2} \rfloor$. Let $C(L)$ be the set of oriented lines obtained from $L = L_1$ as above.

For every $i \geq 1$, define $G_i^-$ and $G_i^+$ as the geometric subgraphs of $G$ induced by $L_i^-$ and $L_i^+$ respectively, and $G_i^-(v_i, v_{i+1})$ and $G_i^+(v_i, v_{i+1})$ as the geometric subgraphs of $G$ induced by $L_i^-(v_i, v_{i+1})$ and $L_i^+(v_i, v_{i+1})$, respectively.

We show there is a line in $C(L)$ for which induction applies to the corresponding graphs $G_i^-$ and $G_i^+$, giving plane spanning trees $T_i^-$ of $G_i^-$ and $T_i^+$ of $G_i^+$. As $T_i^-$ and $T_i^+$ lie in opposite sides of $L$, their union contains a plane spanning tree of $G$. We analyse several cases.

Case 1. $s(G_i^-) \leq |L_i^-| - 3$ and $s(G_i^+) \leq |L_i^+| - 3$.

By induction there exist plane spanning trees $T_i^-$ of $G_i^-$ and $T_i^+$ of $G_i^+$. Since $T_i^-$ and $T_i^+$ lie in opposite sides of $L_i$ and contain exactly one point in common, the graph $T_i^- \cup T_i^+$ is a plane spanning tree of $G$.

Case 2. $s(G_i^-) \geq |L_i^-| - 2$ and $s(G_i^+) \geq |L_i^+| - 2$.

Clearly $s(G_i^-) + s(G_i^+) \geq (|L_i^-| - 2) + (|L_i^+| - 2) = n - 3 \geq s(G) \geq s(G_i^-) + s(G_i^+)$, This implies $s(G_i^-) = |L_i^-| - 2$, $s(G_i^+) = |L_i^+| - 2$ and that $L_1$ does not cross any disconnected empty triangle of $G$. 
Moreover, all lines $v_j$ do not cross any disconnected empty triangle of $G$ and $L_{j+1}$ crosses a disconnected empty triangle $\Delta xyv_j$ of $G$.

Case 2.1. $v_{j+1} \in L_j^-$.  
In this case $\Delta xyz$ is a disconnected empty triangle of $G_j^-$, see Fig. 2 (left) and Fig. 3. Let $i \geq j + 1$ be the smallest integer such that the axis vertex $v_{i+1}$ of $L_{i+1}$ lies in $L_j^-$. By the choice of $i$, all points $v_{j+1}, v_{j+2}, \ldots, v_i$ lie in $L_j^-$ and therefore $L_i^- = L_{i-1}^- = \cdots = L_j^-$. It follows that $G_i^- = G_{i-1}^- = \cdots = G_j^-$ and that $s(G_i^-) = s(G_{i-1}^-) = \cdots = s(G_j^-)$.

Again by the choice of $i$, $L_{i+1}^- = (L_k^- \setminus \{v_i\}) \cup \{v_{i+1}\}$ for $k = j, j+1, \ldots, i-1$ and therefore $L_i^- = (L_j^- \setminus \{v_j\}) \cup \{v_i\}$. Moreover, all lines $L_i, L_{i-1}, \ldots, L_{j+1}$ cross $\Delta xyz$. This implies $s(G_i^-) \leq s(G_j^-) - 1$ since $\Delta xyz$ is a disconnected empty triangle of $G_j^-$.  

Now consider the line $L(v_i, v_{i+1})$ and notice that $L^-(v_i, v_{i+1}) = L_i^-$ and $L^+(v_i, v_{i+1}) = L_i^+ \cup \{v_{i+1}\}$ because $v_{i+1} \in L_i^-$. Therefore

$$|L^-(v_i, v_{i+1})| = |L_i^-| = |L_j^-| \quad \text{and} \quad |L^+(v_i, v_{i+1})| = |L_i^+| + 1 = |L_j^+| + 1.$$

Also notice that $s(G^-(v_i, v_{i+1})) = s(G_i^-)$ and $s(G^+(v_i, v_{i+1})) = s(G_i^+)$ because no empty triangle of $G$ contained in $L^+(v_i, v_{i+1})$ has $v_{i+1}$ as one of its vertices since $L_j$ does not cross any empty triangle of $G$. Therefore

$$s(G^-(v_i, v_{i+1})) = s(G_i^-) - 1 = (|L_j^-| - 2) - 1 = |L_j^-| - 3 = |L^- (v_i, v_{i+1})| - 3$$

and

$$s(G^+(v_i, v_{i+1})) = s(G_i^+) = |L_i^-| - 2 = (|L^+(v_i, v_{i+1})| - 1) - 2 = |L^+(v_i, v_{i+1})| - 3.$$

By induction, there exist plane spanning trees $T^-$ of $G^-(v_i, v_{i+1})$ and $T^+$ of $G^+(v_i, v_{i+1})$. The theorem follows since $T^+ \cup T^-$ contains a plane spanning tree of $G$.

Case 2.2. $v_{j+1} \in L_j^+$.  
In this case $\Delta xyz$ is a disconnected empty triangle of $G_j^+$, see Fig. 2 (right). The proof is analogous to that of Case 2.1.
Case 3. \( s(G_1^-) \geq |L_1^-| - 2 \) and \( s(G_1^+) \leq |L_1^+| - 3 \).

If for every \( L_j \in C(L) \),
\[
s(G_j^-) \geq |L_j^-| - 2 \quad \text{and} \quad s(G_j^+) \leq |L_j^+| - 3,
\]
then for \( L_m \) in particular, the line in \( C(L) \) parallel to \( L_1 \) with the opposite orientation, we have that
\[
s(G_m^-) \geq |L_m^-| - 2 \quad \text{and} \quad s(G_m^+) \leq |L_m^+| - 3.
\]

If \( n \) is odd, then \( L_1 \) and \( L_m \) are the same line but with opposite orientations, in which case \( L_m^- = L_1^+ \) and \( L_m^+ = L_1^- \). It follows that
\[
|L_1^+| - 2 = |L_1^-| - 2 \leq s(G_1^-) = s(G_m^+) \leq |L_m^+| - 3,
\]
which is not possible.

If \( n \) is even, then \( L_1 \) and \( L_m \) are parallel lines with opposite orientations, with \( L_m \) to the left of \( L_1 \) and with \( |L_1^+| + |L_m^-| = n \). This implies that there are no points between \( L_1 \) and \( L_m \). Therefore every empty triangle of \( G_1^- \) contains points in \( L_m^+ \) and every empty triangle of \( G_m^- \) contains points in \( L_1^+ \). Thus no empty triangle of \( G_1^- \) is also an empty triangle of \( G_m^- \), see Fig. 4.

It follows that \( s(G_1^-) + s(G_m^-) \leq s(G) \) which is also a contradiction since
\[
s(G) \leq n - 3 < n - 2 = |L_1^-| - 2 + |L_m^-| - 2 \leq s(G_1^-) + s(G_m^-).
\]

Therefore, there exists \( L_k \in C(L) \) such that
\[
s(G_k^-) \geq |L_k^-| - 2 \quad \text{and} \quad s(G_k^+) \leq |L_k^+| - 3,
\]
while
\[
s(G_{k+1}^-) \leq |L_{k+1}^-| - 3 \quad \text{or} \quad s(G_{k+1}^+) \geq |L_{k+1}^+| - 2.
\]

Since \( L_{k+1}^- = L_k^- \) or \( L_{k+1}^+ = L_k^+ \), it must happen that either
\[
s(G_{k+1}^-) \leq |L_{k+1}^-| - 3 \quad \text{and} \quad s(G_{k+1}^+) \leq |L_{k+1}^+| - 3
\]
or
\[
s(G_{k+1}^-) \geq |L_{k+1}^-| - 2 \quad \text{and} \quad s(G_{k+1}^+) \geq |L_{k+1}^+| - 2,
\]
which are Case 1 and Case 2, respectively.

Case 4. \( s(G_1^-) \leq |L_1^-| - 3 \) and \( s(G_1^+) \geq |L_1^+| - 2 \).

As above, let \( L_m \) be the line in \( C(L) \) parallel to \( L_1 \) with opposite orientation. If \( n \) is odd, then \( L_m^- = L_1^+ \) and \( L_m^+ = L_1^- \). Therefore \( s(G_m^-) \geq |L_1^-| - 2 \) and \( s(G_m^+) \leq |L_1^+| - 3 \) which is Case 3.

For \( n \) even additional considerations are needed. For the sake of completeness we include the entire proof for this subcase.

If \( s(G_j^-) \leq |L_j^-| - 3 \) and \( s(G_j^+) \geq |L_j^+| - 2 \) for every \( L_j \in C(L) \), then, \( s(G_m^-) \leq |L_m^-| - 3 = l - 2 \) and \( s(G_m^+) \geq |L_m^+| - 2 \). As \( L_1^- \subset L_m^- \) and \( L_1^+ \subset L_m^+ \), we have,
\[
|L_1^+| - 2 \leq s(G_1^+) \leq s(G_m^+) \leq |L_m^+| - 2 \quad \text{and} \quad |L_m^-| - 2 \leq s(G_m^-) \leq s(G_1^-) \leq |L_1^-| - 3 = |L_1^+| - 2
\]
which implies \( s(G_1^+) = s(G_1^-) = s(G_m^+) = s(G_m^-) \), since \( |L_1^+| = |L_m^-| \).

Fig. 4. No empty triangle of \( G \) is contained in \( L_1^- \cap L_m^- \).
It follows that no disconnected empty triangle $\Delta xyz$ of $G^+$ has $v_1$ as one of its vertices, otherwise $L_m$ must cross $\{x, y, z\}$ in which case $s(G^+_m) < s(G^-)$ because $G^+_m$ is a subgraph of $G^-$. By our assumption, the same argument can be applied to every line $L_j$ in $C(L)$ and therefore for each graph $G^-_j$, no disconnected empty triangle of $G^-_j$ has $v_j$ as one of its vertices.

To reach a contradiction consider any disconnected empty triangle $\Delta xyz$ of $G^+$. As $L_m$ is parallel to $L_1$ and to the left of $L_1$, then $\Delta xyz$ is also a disconnected empty triangle of $G^-_m$ and therefore $\Delta xyz$ lies to the right of $L_1$ and to the left of $L_m$. By Lemma 2, there is a line $L_t$ in $C(L)$ with $1 < t < m$ such that $\Delta xyz$ is a disconnected empty triangle of $G^+_t$ and one of its vertices is precisely $v_t$, which is the contradiction, see Fig. 5.

As in Case 3, there is a line $L_k$ in $C(L)$ such that

$$s(G^-_k) \leq |L_k^-| - 3 \quad \text{and} \quad s(G^+_k) \geq |L_k^+| - 2,$$

while

$$s(G^-_{k+1}) \geq |L_{k+1}^-| - 2 \quad \text{or} \quad s(G^+_k) \leq |L_{k+1}^+| - 3.$$

Again, since $L_{k+1}^- = L_k^-$ or $L_{k+1}^+ = L_k^+$, it must happen that either

$$s(G^-_{k+1}) \leq |L_{k+1}^-| - 3 \quad \text{and} \quad s(G^+_k) \leq |L_{k+1}^+| - 3,$$

or

$$s(G^-_{k+1}) > |L_{k+1}^-| - 3 \quad \text{and} \quad s(G^+_k) > |L_{k+1}^+| - 3,$$

which are Case 1 and Case 2, respectively. This ends the proof of Theorem 1.

3. Final remark

For $n \geq 5$, let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of a regular $(n-1)$-gon and let $w$ be a point closed to $v_{n-1}$ and in the interior of the triangle $\Delta v_{n-2} v_{n-1}$. Denote by $R_n$ and $R_n^c$ the plane path $v_1, v_2, \ldots, v_{n-1}, w$ and its complement, respectively. The geometric graph $R_n^c$ is such that $s(R_n^c) = n - 3$ and both graphs $R_n$ and $R_n^c$ contain plane spanning trees. This shows that Theorem 1 is not (at least not an immediate) consequence of the result by Károlyi et al. mentioned above.

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