GLOBAL CLF STABILIZATION OF NONLINEAR SYSTEMS. PART I:
A GEOMETRIC APPROACH—COMPACT STRICTLY CONVEX CVS*

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Abstract. The aim of this paper is to design regular feedback controls for the global asymptotic stabilization (GAS) of affine control systems with compact (convex) control value sets (CVS) \( U \subseteq \mathbb{R}^n \) with \( 0 \in \text{int}U \), in the framework of Artstein–Sontag’s control Lyapunov function (CLF) approach. Convex analysis allows us to reveal the intrinsic geometry involved in the CLF stabilization problem, and to solve it, if an optimal control \( \mathcal{Z}(x) \) exists. The study of its existence, uniqueness, and continuity, in terms of properties of \( U \), yields that \( \mathcal{Z}(x) \) is gradient-based, and leads to the generic class \( \mathcal{U}(\mathbb{R}^n) \) of compact strictly convex CVS. Moreover, higher regularity is attained via the geometry (curvature) of \( U \) (illustrated for the \( p, r \)-weighted balls). However, since \( \mathcal{Z}(x) \) is singular, we consider a general form of admissible feedback controls for the GAS of a system, provided a CLF is known. For \( U \in \mathcal{U}(\mathbb{R}^n) \), we design an explicit formula for suboptimal admissible controls, hence generically solving the synthesis problem entailed by Artstein’s theorem. Finally, for a dense class of CVS, if we assume smoothness on the system and the CLF, we obtain an explicit formula for practically smooth feedback controls. The results are illustrated with the limit-cycle suppression of a system in \( \mathbb{R}^1 \) via admissible controls.

Key words. constrained control, nonlinear system, global stabilization, control Lyapunov function, convexity, curvature

AMS subject classifications. 93C10, 93D15, 93B50, 52A41, 93B29

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1. Introduction. Consider the multiple input affine control system

\[
\dot{x} = f(x) + \sum_{j=1}^{m} a_j g_j(x),
\]

where \( x \in \mathbb{R}^n \), \( f, g_j : \mathbb{R}^n \to \mathbb{R}^n \), for \( j = 1, \ldots, m \), are regular vector fields. Here, the word regular means continuous, Lipschitz, of class \( C^s(\mathbb{R}^n) \ (s \geq 1) \), smooth, et cetera. We shall assume that \( f(0) = 0 \). A control value set (CVS) is any convex set \( U \subseteq \mathbb{R}^n \), \( u = (u_1, \ldots, u_m)^T \in U \), and \( ^T \) denotes transposition. We denote by \( \mathbb{K}^U(\mathbb{R}^n) \) the class of all compact CVS \( U \) with \( 0 \in \text{int}U \). An admissible feedback control is any regular function \( \bar{u} : \mathbb{R}^n \to U \). The control design problem is known as the synthesis problem.

Our aim is to address the synthesis of admissible controls \( \bar{u}(x) \), with \( U \in \mathbb{K}^0(\mathbb{R}^n) \), for the global asymptotic stabilization (GAS) of (the rest point 0 of) system (1.1).

In control theory, a control Lyapunov function (CLF) \( V(x) \) is a generalization of the notion of Lyapunov function \( V(x) \) used on the stability analysis of a system of ordinary differential equations \( dx/dt = f(x) \) (or open-loop system: (1.1) with \( u \equiv 0 \)). We say that \( V : \mathbb{R}^n \to [0, \infty) \) is a global strict Lyapunov function for this system, if and only if (i) it is a positive definite \( (V(0) = 0 \text{ and } V(x) > 0, x \neq 0) \) and proper \( (V^{-1}(c) \text{ is compact } \forall c \geq 0, \text{ or equivalently radially unbounded}) \) \( C^s(\mathbb{R}^n) \) function \( (s \geq 1) \), such that

\[
\forall x \neq 0, \ V(x) = L_x V(x) := \nabla V(x) f(x) < 0.
\]

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