Group reflection and precompact paratopological groups

Abstract
We construct a precompact completely regular paratopological Abelian group $G$ of size $(2^\omega)^+$ such that all subsets of $G$ of cardinality $\leq 2^\omega$ are closed. This shows that Protasov’s theorem on non-closed discrete subsets of precompact topological groups cannot be extended to paratopological groups. We also prove that the group reflection of the product of an arbitrary family of paratopological (even semitopological) groups is topologically isomorphic to the product of the group reflections of the factors, and that the group reflection, $H_\ast$, of a dense subgroup $H$ of a paratopological group $G$ is topologically isomorphic to a subgroup of $G_\ast$.

Keywords
precompact • pseudocompact • group reflection • paratopological group

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1. Introduction

It follows from Protasov’s theorem in [8] that every infinite precompact Hausdorff topological group contains a countable discrete non-closed subset. In fact, the main theorem in [8] states that if $A$ is an infinite precompact subset of a Hausdorff topological group $G$, then the set $AA^{-1}$ contains a countable discrete subset $D$ such that $e \notin D \setminus D$, where $e$ is the neutral element of $G$. It is natural, therefore, to find out whether a similar fact is valid for paratopological groups, i.e., groups with a topology in which multiplication is jointly continuous (a survey on paratopological groups can be found in [10]). This question appeared explicitly in [1, Problem 3.7.3] and [10, Problem 4.6].

In Theorem 6 we answer the question in the negative by constructing a dense subgroup $G$ of $\mathbb{K}^\lambda$, with $\lambda = (2^\omega)^+$, such that all subsets of $G$ of cardinality $\leq 2^\omega$ are closed. Here $\mathbb{K}$ denotes the unit circle endowed with the Sorgenfrey topology. Since the paratopological groups $\mathbb{K}$ and $\mathbb{K}^\lambda$ are precompact and so are dense subgroups of precompact paratopological groups (see Lemma 1), the group $G$ is precompact as well.

An interesting additional property of the paratopological group $G$ constructed in Theorem 6 is that the group reflection of $G$, say, $G_\ast$, is a pseudocompact topological group. This is why we pay much attention to the study of the group reflection of paratopological groups. By Proposition 1, for a dense subgroup $H$ of a saturated paratopological group $L$, the group reflection $H_\ast$ of $H$ inherits its topology from the topological group $L_\ast$. The density of $H$ in $L$ is essential (see Remark 4). However, Proposition 1 remains valid without the additional assumption on $L$, i.e., one can drop ‘saturated’ there. This is established in Theorem 12, in the end of the article.

In Lemma 5 we establish the validity of the formula $\left(\prod_{i \in I} G_i\right)_\ast \cong \prod_{i \in I}(G_i)_\ast$ for a family $\{G_i : i \in I\}$ of saturated paratopological groups. Afterwards, in Theorem 10, this result is extended to products of arbitrary semitopological groups.

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Precompact paratopological groups

Since the paratopological group $G$ in Theorem 6 is a dense subgroup of the product $\mathbb{K}^\omega$ of precompact (hence saturated) paratopological groups, Proposition 1 and Lemma 5 enable us to conclude that $G$ is a topological subgroup of the topological group $(\mathbb{K}^\omega)^1 \cong (\mathbb{K}_c)^1 \cong \mathbb{T}^\omega$, where $\mathbb{T}$ is the unit circle group with its usual topology. In turn, this fact is crucial for establishing that $G$ is pseudocompact.

As usual, we call a paratopological group $G$ precompact if for every neighborhood $U$ of the neutral element in $G$, there exists a finite subset $F$ of $G$ such that $FU = G = UF$. The group $G$ is said to be saturated if $U^{-1}$ has a nonempty interior, for each neighborhood $U$ of the neutral element in $G$. The standard examples of saturated paratopological groups are the Sorgenfrey line $\mathbb{S}$ and the group $\mathbb{K} = \mathbb{S}/\mathbb{Z}$. By [9, Proposition 2.1], precompact paratopological groups are saturated. The group reflection of a paratopological group $G$ is the underlying group $G$ endowed with the finest topological group topology weaker than the original topology of $G$.

A semitopological group is a group with a topology invariant with respect to translations. In other words, the left and right translations in the group are homeomorphisms. It is clear that every paratopological group is a semitopological group.

The spaces we consider are not assumed to satisfy any separation axioms if the otherwise is not specified explicitly.

2. Examples

The following result extends [1, 5.1.C] to precompact paratopological groups.

Lemma 1.

A dense subgroup of a precompact paratopological group is precompact.

Proof. Let $H$ be a dense subgroup of a precompact paratopological group $G$. Take an arbitrary neighborhood $U$ of the neutral element $e$ in $G$. First we show that $H$ contains a finite subset $F$ such that $G = UF$. Choose an open neighborhood $V$ of $e$ such that $V^2 \subseteq U$. Since $G$ is precompact, there exists a finite subset $C$ of $G$ such that $G = VC$. For every $x \in C$, take an element $h_x \in H \cap x^{-1}V$ and put $F = \{h_x^{-1} : x \in C\}$. Clearly $F$ is a finite subset of $H$. It follows from $h_x \in x^{-1}V$ that $x \in Vh_x^{-1}$, whence $Vx \subseteq V^2h_x^{-1} \subseteq Uh_x^{-1}$, for each $x \in C$. We conclude, therefore, that $G = VC \subseteq UF$, i.e., $UF = G$.

Finally, suppose that $H$ is an open neighborhood of $e$ in $H$. Take an open set $U$ in $G$ such that $W = U \cap H$. We have just shown that $H$ contains a finite set $F$ such that $G = UF$. Since $H$ is a subgroup of $G$, our choice of $U$ implies that $H = WF$. A similar argument shows that $H$ contains a finite subset $F'$ such that $H = F'W$. Hence $H$ is precompact.

In what follows the unit circle $\mathbb{T}$ will be written multiplicatively. When endowed with the Sorgenfrey topology, it will be denoted by $\mathbb{K}$. Clearly $\mathbb{K}$ is a precompact paratopological Abelian group of size $\omega$. The proof of our second auxiliary result is completely analogous to the proof of a similar fact for the Sorgenfrey line and hence is omitted.

Lemma 2.

The subgroup $D = \{(x, y) \in \mathbb{K}^2 : x \cdot y = 1\}$ of $\mathbb{K}^2$ is closed and discrete in $\mathbb{K}^2$.

Lemma 3.

Let $G$ be a group, $\tau$ be a semitopological group topology on $G$, and $\sigma$ a topological group topology on $G$. If $\tau \subseteq \sigma$ and $\tau$ is a $\pi$-base for $\sigma$, then $\tau = \sigma$.

Proof. Let $e$ be the neutral element of $G$. Since both $\tau$ and $\sigma$ are invariant under translations in $G$, it suffices to show that for every $U \in \sigma$ with $e \in U$, there exists $V \in \tau$ such that $e \in V \subseteq U$. Take an arbitrary $U \in \sigma$ with $e \in U$ and choose $W \in \sigma$ such that $e \in W$ and $WW^{-1} \subseteq U$. Since $\tau$ is a $\pi$-base for $\sigma$, there exists a nonempty set $O \in \tau$ such that $O \subseteq W$. Then the set $V = OO^{-1} = \bigcup\{Ox^{-1} : x \in O\}$ is open in $(G, \tau)$, $e \in V$, and $V = OO^{-1} \subseteq WW^{-1} \subseteq U$. This completes the proof.

It is worth mentioning that one cannot interchange the topologies $\tau$ and $\sigma$ in the formulation of Lemma 3. Indeed, let $\tau$ be the Sorgenfrey topology of the circle group $\mathbb{T}$, and $\sigma$ be the usual topology of the circle group $\mathbb{T}$. Then $\mathbb{K} = (\mathbb{T}, \tau)$ is a paratopological group, $(\mathbb{T}, \sigma)$ is a topological group, $\sigma \subseteq \tau$, $\sigma$ is a $\pi$-base for $\tau$, but $\sigma \neq \tau$.  

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Proposition 1.  
Let $H$ be a dense subgroup of a saturated paratopological group $G$. Then the group reflection $H_\ast$ of $H$ is a topological subgroup of the topological group $G_\ast$.

Proof.  
Denote by $\tau$ and $\tau_\ast$ the topologies of the groups $G$ and $G_\ast$, respectively. Then $\tau_\ast \subseteq \tau$. By [2, Theorem 5], $\tau_\ast$ is a $\pi$-base for $\tau$. Let $\sigma = \tau \vert H$. The topology $\sigma_\ast$ of $H_\ast$ is coarser than $\sigma$. Further, it is clear that $\tau_\ast \vert H \subseteq \sigma_\ast$, and the density of $H$ in $G$ implies that $\tau_\ast \vert H$ is a $\pi$-base for $\tau \vert H = \sigma$. Since $\tau_\ast \vert H \subseteq \sigma \subseteq \sigma$, we see that $\tau_\ast \vert H$ is a $\pi$-base for $\sigma_\ast$. Taking into account that both $\tau_\ast \vert H$ and $\sigma_\ast$ are topological group topologies on $H$ and applying Lemma 3, we conclude that $\tau_\ast \vert H = \sigma_\ast$. This means that $H_\ast$ is a topological subgroup of $G_\ast$. \hfill $\blacksquare$

Remark 4.  
In Proposition 1, one cannot drop the assumption of the density of $H$ in $G$. Indeed, let $S$ be the Sorgenfrey line and $G = S^2$. Then $S$ and $G$ are saturated paratopological groups. It is clear that

$$D = \{(x, y) \in S^2 : x + y = 0\}$$

is a closed discrete subgroup of $G$, so $D_\ast = D$ is a discrete topological group. However, it is easy to see that $G_\ast = \mathbb{R}^2$, so $D$ considered with the topology inherited from $G_\ast$ is topologically isomorphic to the real line $\mathbb{R}$.

We will see in Theorem 12, however, that Proposition 1 remains valid without the assumption that $G$ is saturated. Since the proof of Theorem 12 is considerably longer than our argument in the proof of Proposition 1, we place the theorem in the end of the article.

For the proof of Theorem 6 we need the following fact which will be extended to products of arbitrary paratopological (even semitopological) groups in Theorem 10.

Lemma 5.  
Let $\Pi = \prod_{i \in I} G_i$ be a product of saturated paratopological groups. Then $\Pi_\ast$ is topologically isomorphic to $\prod_{i \in I}(G_i)_\ast$.

Proof.  
Let $\tau$ and $\tau_\ast$ be the topologies of $\Pi$ and $\Pi_\ast$, respectively. Take an arbitrary nonempty set $U \in \tau$. Since $\tau_\ast \subseteq \tau$, we can find a nonempty canonical open set $V \in \tau$ such that $V \subseteq U$. Suppose that $V = \bigcap_{1 \leq k \leq n} \pi_k^{-1}(V_k)$, where $n$ is a positive integer, $i_k \in I$, $V_k$ is open in $G_{i_k}$, and $\pi_k$ is the projection of $\Pi$ to $G_{i_k}$ for each $k$ with $1 \leq k \leq n$. Since the groups $G_{i_k}$ are saturated, [2, Theorem 5] implies that the topology of $G_{i_k}$ is a $\pi$-base for $G_{i_k}$ for $k = 1, \ldots, n$.

For every $k \leq n$, take a nonempty open subset $W_k$ of $G_{i_k}$, such that $W_k \subseteq V_k$. Then the set $W = \bigcap_{1 \leq k \leq n} \pi_k^{-1}(W_k)$ is nonempty, open in $\prod_{i \in I}(G_i)_\ast$, and satisfies $W \subseteq V \subseteq U$. Therefore, the topology of $\prod_{i \in I}(G_i)_\ast$, say, $\sigma$ is a $\pi$-base for $\Pi_\ast$.

Since $\sigma \subseteq \tau_\ast \subseteq \tau$, we see that $\sigma$ is a $\pi$-base for $\tau$. Taking into account that $\sigma$ and $\tau_\ast$ are topological group topologies and applying Lemma 3, we conclude that $\sigma = \tau_\ast$. In other words, $\Pi_\ast$ is topologically isomorphic to $\prod_{i \in I}(G_i)_\ast$. \hfill $\blacksquare$

Since precompact paratopological groups are saturated by [9, Proposition 2.1], the following corollary is immediate:

Corollary 2.  
Let $\Pi = \prod_{i \in I} G_i$ be a product of precompact paratopological groups. Then $\Pi_\ast$ is topologically isomorphic to $\prod_{i \in I}(G_i)_\ast$.

The theorem below is one of the main results of the article. We denote by $c$ the power of the continuum, i.e., $c = 2^\omega$.

Theorem 6.  
There exists a precompact completely regular paratopological Abelian group $G$ of size $c^+$ such that all subsets of $G$ of cardinality $\leq c$ are closed. In addition, the group reflection $G_\ast$ of $G$ is pseudocompact, so $G$ admits a continuous isomorphism onto a pseudocompact topological group.
Proof. Our aim is to construct the group $G$ with the required combination of properties as a dense subgroup of $K^\lambda$, where $\lambda = \omega^\omega$. Then, by Lemma 1, $G$ will be precompact.

We will define by recursion a dense subset $Y = \{y_\alpha : \alpha < \lambda\}$ of $K^\lambda$ and then put $G = \langle Y \rangle$. In fact, $Y$ will have a stronger property—we will construct $Y$ to satisfy $\pi_\alpha(Y) = K^\alpha$ for each $\alpha \in [\lambda]^\omega$, where $\pi_\alpha$ is the projection of $K^\lambda$ to $K^\alpha$. As usual, $[\lambda]^\omega$ stands for the family of all countable subsets of $\lambda$.

First we have to introduce some notation. Since the set

$$\bigcup\{[\lambda]^\omega : B \in [\lambda]^\omega, B \neq \emptyset\}$$

is of size $\lambda$, we enumerate it as $\{x_\alpha : 0 < \alpha < \lambda\}$. For every $\alpha < \lambda$ distinct from 0, take $B_\alpha \in [\lambda]^\omega$ such that $x_\alpha \in K^{B_\alpha}$.

We can additionally choose the above enumeration to satisfy $B_\alpha \subseteq \alpha$ if $0 < \alpha < \lambda$.

Let $\delta_0 = 0$. For every $\alpha < \lambda$ we put $\delta_{\alpha + 1} = \delta_\alpha + 2$. For a limit ordinal $\alpha < \lambda$ distinct from 0, we put $\delta_\alpha = \sup_{\beta < \alpha} \delta_\beta$.

It is easy to verify that, in the latter case, $\delta_\alpha = \alpha$. Also, $\alpha \leq \delta_\alpha$ for each $\alpha < \lambda$.

Finally, let $D$ be an independent subset of the circle group $T$ such that $|D| = \omega$. For every ordinal $\alpha$ with $0 < \alpha < \lambda$, we fix an injective mapping $h_\alpha : \alpha \to D$.

At a step $\alpha$ of our construction, we will define the values $y_\beta(\nu) \in K$ for all $\beta < \alpha$ and $\nu < \delta_\alpha + 1$ satisfying the following conditions:

(i) $y_\beta(\nu) = x_\beta(\nu)$ for each $\nu \in B_\beta$;

(ii) $y_\beta(\delta_\alpha) \cdot y_\beta(\delta_\alpha + 1) = 1$ for each $\beta < \alpha$;

(iii) the set $\{y_\beta(\delta_\alpha) : \beta < \alpha\}$ is independent in $T$.

Condition (i) means that each $y_\alpha$ extends $x_\alpha$, while (ii) and (iii) guarantee that "small" subsets of $G = \langle Y \rangle$ are closed and discrete.

Let us start the construction of the set $Y \subseteq K^\lambda$. First we put $y_0(0) = y_0(1) = 1$. Note that (i)–(iii) are vacuous at the step 0. Suppose that for some $\alpha$ with $0 < \alpha < \lambda$, we have defined the values $y_\beta(\nu) \in K$ for all $\beta < \alpha$ and $\nu < \delta_\alpha$. Now we have to define $y_\alpha(\nu)$ for each $\nu < \delta_\alpha + 1$ and $y_\beta(\nu)$ for all $\beta < \alpha$ and $\nu \in \{\delta_\alpha, \delta_\alpha + 1\}$. According to (i), let us put $y_\alpha(\nu) = x_\alpha(\nu)$ if $\nu \in B_\alpha$ (note that $B_\alpha \subseteq \alpha < \delta_\alpha$) and $y_\alpha(\nu) = 1$ if $\nu \in \delta_\alpha + 1 \setminus B_\alpha$.

It remains to define the values $y_\alpha(\delta_\alpha)$ and $y_\beta(\delta_\alpha + 1)$ by $y_\alpha(\delta_\alpha) = h_\alpha(\delta_\alpha)$ and $y_\beta(\delta_\alpha + 1) = h_\beta(\delta_\alpha + 1)$ for each $\beta < \alpha$. This finishes our construction of the family $\{y_\beta(\nu) : \beta < \alpha, \nu < \delta_\alpha + 1\}$ satisfying (i)–(iii).

Thus we have defined the set $\{y_\alpha : 0 < \alpha < \lambda\} \subseteq K^\lambda$. We claim that the subgroup $G = \langle Y \rangle$ of $K^\lambda$ is as required. To show that $G$ is precompact it suffices, by Lemma 1, to verify that $G$ is dense in $K^\lambda$. Let us show that $G$ fills all countable subgroups of $K^\lambda$, i.e., $\pi_\alpha(G) = K^\alpha$ for each countable subset $A$ of $\alpha$ and, hence, $G$ is dense in $K^\lambda$. Let $x \in K^\lambda$, for a nonempty countable $A$-set in $K^\lambda$. Then $x = x_\alpha$ for some $\alpha < \lambda$ and $A = B_\alpha$, so (i) implies that $x_\alpha = \pi_\alpha(y_\alpha)$. Therefore, $\pi_\alpha(Y) = \pi_\alpha(G) = K^\alpha$. In particular, $G$ meets every nonempty $\omega_\alpha$-set in $K^\lambda$.

Let us show that all subsets of $G$ of cardinality $\leq \omega$ are closed. Suppose that $C \subseteq G$ and $|C| \leq \omega$. Then there exists $\alpha < \lambda$ such that $C$ is a subset of the group $H_\alpha$ generated by the set $\{y_\beta : \beta < \alpha\}$. According to (ii), we have that $y_\beta(\delta_\alpha) \cdot y_\beta(\delta_\alpha + 1) = 1$ for each $\beta < \alpha$. Therefore, the equality $g(\delta_\alpha) \cdot g(\delta_\alpha + 1) = 1$ holds for each element $g \in H_\alpha$. Let $f = \{\delta_\alpha, \delta_\alpha + 1\}$. We see that the projection of $H_\alpha$ to $K^\lambda$ is contained in the subgroup

$$\mathbb{D}_f = \{(a, b) \in K^\lambda : a \cdot b = 1\}$$

of $K^\lambda$ and that $\mathbb{D}_f$ is discrete in $K^\lambda$ by Lemma 2. In addition, (iii) implies that the restriction of this projection to $H_\alpha$ is a monomorphism. Hence $H_\alpha$ is a discrete subgroup of $G$. Let $C_\alpha$ be the closure of $H_\alpha$ in $K^\lambda$. Notice that $K^\lambda$ is an almost topological group in the sense of [6], i.e., $K$ has a local base $\mathcal{B}$ at the neutral element 1 such that $U \setminus \{1\}$ is open in the usual topology of $T$ for each $U \in \mathcal{B}$. Therefore, by [6, Theorem 3.7], $C_\alpha$ is a subgroup of $K^\alpha$. Since $H_\alpha$ is discrete, we see that $H_\alpha$ is an open dense subgroup of the group $G_\alpha$. Therefore $G_\alpha = H_\alpha$, i.e., $H_\alpha$ is closed in $K^\lambda$ and in $G$. We have thus proved that $H_\alpha$ is a closed and discrete subgroup of $G$. Since $C \subseteq H_\alpha$, we conclude that $C$ is also closed and discrete in $G$.

Finally, we verify that the group reflection of $G$, say, $G_\ast$, is pseudocompact. It follows from Proposition 1 that $G_\ast$ is a toposological subgroup of the group reflection of $K^\lambda$. By Corollary 2, the group reflection of $K^\lambda$ is topologically isomorphic.
to $T'$ since $K \cong T$. Thus $G_\ast$ is a topological subgroup of $T'$. Since $G$ meets every nonempty $G_\ast$-set in $K'$, the group $G_\ast$ has a similar property in the compact group $T'$. Therefore, $G_\ast$ is pseudocompact according to [3, Theorem 1.2]. This completes the proof.

\[ \square \]

**Remark 7.**
The construction in Theorem 6 can be easily modified to produce an example of a precompact completely regular paratopological group $H$ with $|H| = c$ such that all countable subsets of $H$ are closed and discrete.

Since the group reflection of the group $G$ constructed in Theorem 6 is pseudocompact, one can try to strengthen this property of $G$ as follows:

**Problem 8.**
Does there exist an infinite precompact paratopological group $G$ such that all countable subsets of $G$ are closed and the group reflection $G_\ast$ of $G$ is countably compact (or even compact) and Hausdorff?

Let us extend Lemma 5 to products of arbitrary paratopological groups. In fact, we will show that the result is valid even if the factors are semitopological groups. Again, the group reflection of a semitopological group $G$, denoted by $G_\ast$, is the same abstract group $G$ endowed with the finest topological group topology which is coarser than the topology of $G$. We start with the case of finitely many factors:

**Lemma 9.**
Let $\prod = \prod_{1 \leq k \leq n} G_k$ be a product of semitopological groups. Then $\prod_\ast$ is topologically isomorphic to $\prod_{1 \leq k \leq n} (G_k)_\ast$.

**Proof.** It suffices to prove the lemma in the case of two factors, say, $G$ and $H$. So let $\prod = G \times H$. Again, let $\tau$ and $\tau_\ast$ be the topologies of $\prod$ and $\prod_\ast$, respectively. Let also $\sigma$ be the topology of the product $G \times H_\ast$. It is clear that $\sigma \subseteq \tau_\ast$. Denote by $\pi$ the projection of $\prod$ onto the second factor $H$. The kernel of $\pi$ is $N = G \times \{e_H\} \cong G$, where $e_H$ is the neutral element of $H$. We claim that the restrictions of $\sigma$ and $\tau_\ast$ to $N$ coincide. First, we note that $\sigma \upharpoonright N \subseteq \tau_\ast \upharpoonright N$ since $\sigma \subseteq \tau_\ast$. To deduce the converse inclusion, let us identify the groups $G$ and $N$ by means of the natural topological isomorphism, $(x, e_H) \mapsto x$ for $x \in G$. Since $N$ is a subgroup of $\prod$, we have the inclusion $\tau_\ast \upharpoonright N \subseteq (\tau_\ast)_N$, where $(\tau_\ast)_N$ is the topology of the semitopological group $G$. It also follows from the definition of $\sigma$ that $\sigma \upharpoonright N = (\tau_\ast)_N$. Therefore, we conclude that $\tau_\ast \upharpoonright N \subseteq \sigma \upharpoonright N$. This proves the equality $\tau_\ast \upharpoonright N = \sigma \upharpoonright N$.

Given a family $U$ of subsets of $\prod$, we denote by $\pi(U)$ the family $\{\pi(U) : U \in U\}$. Now we claim that $\pi(\tau_\ast) = \pi(\sigma)$. Again, the inclusion $\pi(\sigma) \subseteq \pi(\tau_\ast)$ is evident. It is also clear that $\pi(\tau_\ast)$ is a topological group topology on $H$ coarser than the original topology $\tau_\ast$ of $H$, so $\pi(\tau_\ast) \subseteq (\tau_\ast)_N$. Further, the definition of $\sigma$ implies that $\pi(\sigma) = (\tau_\ast)_N$. Therefore, $\pi(\tau_\ast) \subseteq \pi(\sigma)$. Again, this proves the equality $\pi(\tau_\ast) = \pi(\sigma)$.

Summing up, the topological group topologies $\sigma$ and $\tau_\ast$ on $\prod$ satisfy the conditions $\sigma \subseteq \tau_\ast$, $\sigma \upharpoonright N = \tau_\ast \upharpoonright N$, and $\pi(\tau_\ast) = \pi(\sigma)$. Therefore, by [4, Lemma 1], the topologies $\sigma$ and $\tau_\ast$ coincide. Hence $\prod_\ast \cong G \times H_\ast$. \[ \square \]

**Theorem 10.**
Let $\prod = \prod_{i \in I} G_i$ be the product of an arbitrary family of semitopological groups. Then $\prod_\ast$ is topologically isomorphic to $\prod_{i \in I} (G_i)_\ast$.

**Proof.** Take an arbitrary neighborhood $U$ of the neutral element $e$ in $\prod_\ast$ and choose another neighborhood $O$ of $e$ in $\prod_\ast$ such that $O^2 \subseteq U$. Since the topology $\tau$ of $\prod$ is finer than the topology $\tau_\ast$ of $\prod_\ast$, we can find a canonical open set $W$ in $\prod_\ast$ such that $e \in W \subseteq O$. Suppose that $W = \prod_{k \in I} \pi_k^{-1}(W_k)$, where $\pi_k \in I$, $W_k$ is open in $G_k$, and $\pi_k$ is the projection of $\prod$ to $G_k$ for $k = 1, \ldots, n$. Let $I = \{i_k : 1 \leq k \leq n\}$, $G = \prod_{i_k \in I} G_{i_k}$, and $H = \prod_{i_k \in I} G_{i_k}$. Then $\prod = G \times H$ and $W = p^{-1}(W^*)$, where $W^* = \prod_{i_k \in I} W_{i_k}$ is an open neighborhood of the neutral element in $G$ and $p : G \times H \rightarrow G$ is the
projection. By Lemma 9, there exist open neighborhoods $O_1$ and $O_2$ of the neutral elements in $G_*$ and $H_*$, respectively, such that $O_1 \times O_2 \subseteq O$. Then $V = O_1 \cdot W^*$ is an open neighborhood of the neutral element in $G_*$ and we have:

$$p^{-1}(V) = V \times H = (O_1 \times O_2) \cdot (W^* \times H) \subseteq O \cdot O \subseteq U.$$ 

We apply Lemma 9 once again to find open neighborhoods $V_1, \ldots, V_n$ of the neutral elements in $(G_*)_1, \ldots, (G_*)_n$, respectively, such that $V^* = \prod_{i=1}^n V_i \subseteq V$. Therefore, the canonical open set $p^{-1}(V^*)$ in $\Pi_*$ satisfies

$$e \in p^{-1}(V^*) \subseteq p^{-1}(V) \subseteq U.$$ 

This proves that the topology of $\Pi_*$ and the Tychonoff product topology of $\prod_{i \in I}(G_*)_i$ coincide. \hfill \Box

Finally we generalize Proposition 1 to dense subgroups of arbitrary paratopological groups. This generalization requires an explicit description of a neighborhood base at the neutral element of the group reflection of a paratopological group which is given below. In fact, our description works in the wider class of semitopological groups.

In what follows the group of all permutations of the set $\{1, 2, \ldots, n\}$ is denoted by $S(n)$.

**Lemma 11.**

Let $G$ be a semitopological group with neutral element $e$. To every function $\varphi: \mathbb{N} \to \mathcal{N}(e)$, where $\mathcal{N}(e)$ is the family of open neighborhoods of $e$ in $G$, we assign the set

$$O(\varphi) = \bigcup_{n \in \mathbb{N}} \bigcup_{\pi \in S(2n)} U_{\pi(1)} U_{\pi(2)}^{-1} \cdots U_{\pi(2n-1)} U_{\pi(2n)}^{-1},$$

where $U_i = \varphi(i)$ for each $i \in \mathbb{N}$. Then the family

$$\mathcal{N}_*(e) = \{O(\varphi) : \varphi \in \mathcal{N}(e)^\mathbb{N}\}$$

is a local base at the neutral element of the topological group $G_*$. 

**Proof.** First we claim that the family $\mathcal{N}_*(e)$ constitutes a local base at $e$ for a topological group topology, say, $\sigma$ on $G$. To this end it suffices to verify that the family $\mathcal{N}_*(e)$ satisfies Pontryagin's conditions i)–v) given in [1, Theorem 1.3.12] (see also [7, § 18, Theorem 9]). We reproduce them below:

(i) for every $U \in \mathcal{N}_*(e)$, there is an element $V \in \mathcal{N}_*(e)$ such that $V^2 \subseteq U$;

(ii) for every $U \in \mathcal{N}_*(e)$, there is an element $V \in \mathcal{N}_*(e)$ such that $V^{-1} \subseteq U$;

(iii) for every $U \in \mathcal{N}_*(e)$ and every $x \in U$, there is $V \in \mathcal{N}_*(e)$ such that $Vx \subseteq U$;

(iv) for every $U \in \mathcal{N}_*(e)$ and every $x \in G$, there is $V \in \mathcal{N}_*(e)$ such that $xVx^{-1} \subseteq U$;

(v) for any elements $U, V \in \mathcal{N}_*(e)$, there exists $W \in \mathcal{N}_*(e)$ such that $W \subseteq U \cap V$.

We start with (i). Take an element $U \in \mathcal{N}_*(e)$. Then $V = O(\varphi)$ for some $\varphi \in \mathcal{N}(e)^\mathbb{N}$. Put $U_i = \varphi(i)$ and choose $V_i \in \mathcal{N}(e)$ such that $V_i \subseteq U_{2i-1} \cap U_{2i}$ for each $i \in \mathbb{N}$. Define $\psi \in \mathcal{N}(e)^\mathbb{N}$ by $\psi(i) = V_i$ for $i \in \mathbb{N}$. We claim that $V = O(\psi)$ is as required. Indeed, take arbitrary $n \in \mathbb{N}$ and $\sigma \in S(2n)$. By the definition of $O(\varphi)$ and $O(\psi)$ it suffices to show that

$$\left(\bigvee_{\sigma(1)} V_{\sigma(2)}^{-1} \cdots V_{\sigma(2n-1)} V_{\sigma(2n)}^{-1}\right)^{-1} \subseteq U_{\sigma(1)}^{-1} U_{\sigma(2)}^{-1} \cdots U_{\sigma(2n-1)}^{-1} U_{\sigma(2n)}^{-1}$$

for an appropriately chosen permutation $\sigma \in S(4n)$. Let us put $\pi(i) = 2\sigma(i) - 1$ and $\pi(2n + i) = 2\sigma(i)$ for each $i = 1, \ldots, 2n$. Then $\sigma$ is an element of $S(4n)$ for which the above inclusion is valid. This implies (i).

Notice that the sets in $\mathcal{N}_*(e)$ are symmetric, i.e., $U^{-1} = U$ for each $U \in \mathcal{N}_*(e)$. Hence (ii) is valid trivially.
Let us verify (iii). Take \( \varphi \in \mathcal{N}(e)\) and an element \( x \in O(\varphi) \). Then there exist \( n \in \mathbb{N} \) and \( \pi \in S(2n) \) such that \( x \in U_{n(1)} U_{n(2)}^{-1} \cdots U_{n(2n-1)} U_{n(2n)}^{-1} \). We put \( V_i = U_{i+1} \) for each \( i \in \mathbb{N} \) and define \( \psi \in \mathcal{N}(e)\) by \( \psi(i) = V_i \), where \( i \in \mathbb{N} \). Again, by the definition of \( O(\varphi) \) and \( O(\psi) \), it suffices to show that for every \( m \in \mathbb{N} \) and \( \sigma = S(2m) \), there exists \( \lambda \in S(2m + 2n) \) such that

\[
V_{\sigma(1)} V_{\sigma(2)}^{-1} \cdots V_{\sigma(2m-1)} V_{\sigma(2m)}^{-1} x \subseteq U_{\sigma(1)} U_{\sigma(2)}^{-1} \cdots U_{\sigma(2m-1)} U_{\sigma(2m)}^{-1}.
\]

We define the required permutation \( \lambda \) by \( \lambda(i) = \sigma(i) + 2n \) for \( i = 1, \ldots, 2m \) and \( \lambda(2m + i) = \pi(i) \) for \( i = 1, \ldots, 2n \). A direct verification shows that \( \lambda \in S(2m + 2n) \) and that

\[
V_{\sigma(1)} V_{\sigma(2)}^{-1} \cdots V_{\sigma(2m-1)} V_{\sigma(2m)}^{-1} x \subseteq U_{\sigma(1)} V_{\sigma(2)}^{-1} \cdots U_{\sigma(2m-1)} V_{\sigma(2m)}^{-1} U_{\sigma(2m+1)}^{-1} \cdots U_{\sigma(2m+2n)}^{-1} = U_{\sigma(1)} U_{\sigma(2)}^{-1} \cdots U_{\sigma(2m-1)} U_{\sigma(2m)}^{-1} U_{\sigma(2m+1)}^{-1} \cdots U_{\sigma(2m+2n)}^{-1}.
\]

This implies (iii).

To verify (iv), we take \( x \in G \) and \( U = O(\varphi) \) for some \( \varphi \in \mathcal{N}(e)\), and put \( U_i = \varphi(i) \) for each \( i \in \mathbb{N} \). If \( n \in \mathbb{N} \) and \( \pi \in S(2n) \), then

\[
x\left(U_{n(1)} U_{n(2)}^{-1} \cdots U_{n(2m-1)} U_{n(2m)}^{-1}\right)x^{-1} = \left(x U_{n(1)}^{-1} x^{-1}\right) \cdots \left(x U_{n(2m-1)}^{-1} x^{-1}\right) \left(x U_{n(2m)}^{-1} x^{-1}\right).
\]

Therefore, for every \( i \in \mathbb{N} \), we choose \( V_i \in \mathcal{N}(e) \) satisfying \( x V_i x^{-1} \subseteq U_i \) and define \( \psi \in \mathcal{N}(e)\) by \( \psi(i) = V_i \) for each \( i \in \mathbb{N} \). The above equality shows that \( xO(\psi)x^{-1} \subseteq O(\varphi) \). The verification of (iv) is elementary and we omit it.

Thus there exists a topological group topology \( \sigma \) on \( G \) such that \( \mathcal{N}_\sigma(e) \) is a local base at \( e \) for \( (G, \sigma) \). It follows from the definition of the sets \( O(\varphi) \) that they all open in \( G \). We conclude, therefore, that \( \sigma \) is coarser than the original topology \( \tau \) of \( G \). This implies in turn that \( \sigma \subseteq \tau \), where \( \tau \) is the topology of \( G \). So it remains to show that \( \tau \subseteq \sigma \).

Let \( W \) be an open neighborhood of \( e \) in \( G \). Since \( G \) is a topological group, we can define a sequence \( \{W_i : i \in \mathbb{N}\} \) of open symmetric neighborhoods of \( e \) in \( G \) such that \( W_0 \subseteq W \) and \( W_{i+1} \subseteq W_i \) for each \( i \in \mathbb{N} \). It follows from [1, Lemma 7.2.6] that for every integer \( n \in \mathbb{N} \) and every permutation \( \pi \in S(n) \), we have the inclusion

\[
W_{n(1)} W_{n(2)} \cdots W_{n(n)} \subseteq W.
\]

Since \( \tau \) is finer than \( \tau_n \), we choose, for every \( i \in \mathbb{N} \), an open neighborhood \( U_i \) of \( e \) in \( G \) such that \( U_i \subseteq W_i \). Let \( \varphi \in \mathcal{N}(e)\) be defined by \( \varphi(i) = U_i \) for \( i \in \mathbb{N} \). Our choice of \( \varphi \) and the symmetry of the sets \( W_i \)'s together imply that \( O(\varphi) \subseteq W \). We have thus proved that every neighborhood of the neutral element in \( G \), contains an element of \( \mathcal{N}_\sigma(e) \). Therefore, \( \tau \subseteq \sigma \). This implies that \( \sigma = \tau \), and completes the proof of the lemma.

\[\square\]

**Theorem 12.**

If \( H \) is a dense subgroup of an arbitrary paratopological group \( G \), then \( H \) carries the topology inherited from \( G \).

**Proof.** It is clear that the topology of \( H \) is finer than the topology inherited by \( H \) from \( G \). Hence, according to Lemma 11, it suffices to show that for every \( \varphi \in \mathcal{N}_H(e)\), there exists \( \psi \in \mathcal{N}_C(e)\) such that \( O(\psi) \cap H \subseteq O(\varphi) \). Here \( \mathcal{N}_H(e) \) and \( \mathcal{N}_C(e) \) stand for the families of open neighborhoods of the neutral element \( e \) in the paratopological groups \( H \) and \( G \), respectively.

Suppose that \( \varphi \in \mathcal{N}_H(e)\) and put \( U_i = \varphi(i) \), for each \( i \in \mathbb{N} \). Choose sequences \( \{V_i : i \in \mathbb{N}\} \subseteq \mathcal{N}_C(e) \) and \( \{W_i : i \in \mathbb{N}\} \subseteq \mathcal{N}_C(e) \) such that \( V_i \cap H = U_i \) and \( W_i \subseteq V_i \), for each \( i \in \mathbb{N} \). Let us define \( \psi \in \mathcal{N}_C(e) \) by \( \psi(i) = W_i \) for all \( i \in \mathbb{N} \). We claim that \( O(\psi) \cap H \subseteq O(\varphi) \). The latter inclusion will follow if we show that

\[
(W_{n(1)} W_{n(2)}^{-1} \cdots W_{n(2m-1)} W_{n(2m)}^{-1}) \cap H \subseteq U_{n(1)} U_{n(2)}^{-1} \cdots U_{n(2m-1)} U_{n(2m)}^{-1}.
\]
for all $n \in \mathbb{N}$ and $\pi \in S(2n)$. Given a permutation $\pi \in S(2n)$, we can re-enumerate the sets $U_i$’s and $W_i$’s (taking $U'_i = U_{\pi(i)}$ and $W'_i = W_{\pi(i)}$ for each $i \leq 2n$), thus reducing (1) to the simpler inclusion

$$\left\{ W_1 W_2^{-1} \cdots W_{2n-1} W_{2n}^{-1} \right\} \cap H \subseteq U_1 U_2^{-1} \cdots U_{2n-1} U_{2n}^{-1}. \quad (2)$$

In other words, it suffices to verify (1) for the identity permutation $\pi$. By technical reasons, we will prove the following claim which is slightly stronger than (2).

**Claim.** Let $n \in \mathbb{N}$ and suppose that $W_1, \ldots, W_{2n}$ are elements of $N_G(e)$. Then, for every set $O \in N_G(e)$ and every choice $w_i \in W_i$ for $i = 1, \ldots, 2n$ satisfying $w_1 w_2^{-1} \cdots w_{2n-1} w_{2n}^{-1} \in H$, there exist elements $u_i \in H \cap w_i O$, for $i = 1, \ldots, 2n$, such that

$$u_1 u_2^{-1} \cdots u_{2n-1} u_{2n}^{-1} = w_1 w_2^{-1} \cdots w_{2n-1} w_{2n}^{-1}.$$

Inclusion (2) follows if we take $O = \bigcap_{i=1}^{2n} W_i$ in the above Claim. We will prove Claim by induction on $n$. Let $n = 1$, $O \in N_G(e)$, and suppose that elements $w_1 \in W_1$ and $w_2 \in W_2$ satisfy $w_1 w_2^{-1} = h \in H$. Since $H$ is dense in $G$, we can find an element $u_1 \in W_1 O \cap H$. Take $x \in O$ such that $w_1 x = u_1$ and put $u_2 = w_2 x$. Then $u_2 \in W_2 O$ and

$$h = w_1 w_2^{-1} = w_1 x (w_2 x)^{-1} = u_1 u_2^{-1}.$$

The above equality implies that $u_2 = h^{-1} u_1$, so $u_2 \in H$. This proves Claim for $n = 1$.

Suppose that our claim is valid for some $n \in \mathbb{N}$ and take $w_i \in W_i$ for $i = 1, 2, \ldots, 2n+2$ such that $w_1 w_2^{-1} \cdots w_{2n+1} w_{2n+2} = h \in H$. Choose $O_1, O_2 \in N_G(e)$ satisfying $O_1 \subseteq O$ and $z^{-1} O_1 z \subseteq O_2$ for each $z \in \{ w_{2n}, w_{2n+1} \}$. Since $H$ is dense in $G$, we can find an element $u_{2n+2} \in H \cap w_{2n+2} O_1$. Take $x \in O_1$ such that $u_{2n+2} = w_{2n+2} x$ and let $v_{2n+1} = w_{2n+1} x$. Then $v_{2n+1} \in w_{2n+1} O_1$ and $v_{2n+1} u_{2n+2}^{-1} = w_{2n+1} w_{2n+2}^{-1}$, so

$$h = w_1 w_2^{-1} \cdots w_{2n+1} w_{2n+2} = (w_1 w_2^{-1} \cdots w_{2n-1} w_{2n}^{-1}) v_{2n+1} u_{2n+2}^{-1}.$$ 

Now we repeat the above argument with $v_{2n+1}$ in place of $w_{2n+2}$. First, we find $y \in O_2$ and $u_{2n+1} \in H$ such that $y v_{2n+1} = u_{2n+1}$. It is clear that

$$u_{2n+1} = y w_{2n+1} x \in O_2 w_{2n+1} O_1 = w_{2n+1} (w_{2n+1} O_2 w_{2n+1}) O_1 \subseteq w_{2n+1} O_1 O_1 \subseteq w_{2n+1} O.$$

Let $v_{2n} = y w_{2n}$. Then

$$h = w_1 w_2^{-1} \cdots w_{2n+1} w_{2n+2} = (w_1 w_2^{-1} \cdots w_{2n-1} v_{2n}^{-1}) u_{2n+1} u_{2n+2}^{-1}. \quad (3)$$

Therefore,

$$w_1 w_2^{-1} \cdots w_{2n+1} v_{2n+2}^{-1} = h u_{2n+2} u_{2n+1}^{-1}, \quad (4)$$

where $h_1 = h u_{2n+2} u_{2n+1}^{-1} \in H$. It remains to apply the inductive assumption to the product $w_1 w_2^{-1} \cdots w_{2n-1} v_{2n}^{-1} = h_1$ and find elements $u_1, \ldots, u_{2n}$ with $u_i \in H \cap w_i O_1$ for $i = 1, \ldots, 2n-1$ and $u_{2n} \in H \cap v_{2n} O_1$ satisfying

$$w_1 w_2^{-1} \cdots w_{2n-1} v_{2n}^{-1} = u_1 u_2^{-1} \cdots u_{2n-1} u_{2n}^{-1}. \quad (5)$$

Notice that $u_i \in w_i O_1 \subseteq w_i O$ for each $i = 1, \ldots, 2n-1$ and

$$u_{2n} \in v_{2n} O_1 = y w_{2n} O_1 \subseteq O_2 w_{2n} O_1 = w_{2n} (w_{2n} O_2 w_{2n}) O_1 \subseteq w_{2n} O_1 O_1 \subseteq w_{2n} O.$$

It now follows from (3), (4), and (5) that

$$w_1 w_2^{-1} \cdots w_{2n+1} w_{2n+2} = u_1 u_2^{-1} \cdots u_{2n+1} u_{2n+2}^{-1},$$

where $u_i \in H \cap w_i O$ for each $i \leq 2n + 2$. This completes the proof of Claim and implies the conclusion of the theorem. □

**Problem 13.**

Is Theorem 12 valid for dense subgroups of semitopological groups?
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References


