The Implicit Function Theorem: History, Theory, and Applications

Steven G. Krantz and Harold R. Parks

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MAA REVIEW

[Reviewed by Felipe Zaldivar, on 03/15/2013]

The implicit function theorem in its various guises (the inverse function theorem or the rank theorem) is a gem of geometry, taking this term in its broadest sense, encompassing analysis, both real and complex, differential geometry and topology, algebraic and analytic geometry. Taking the above statement for granted, the publication in 2002 of the original edition of the book under review and its reprint in the Modern Birkhäuser Classics in 2013, is a welcome opportunity to look at the history of the various formulations of this theorem, from the elementary (a version for real-valued functions of several real variables), to the classical formulation for $C^k$–functions between Euclidean spaces, and to the Nash-Moser version. The book under review balances the historical analyses with presentations and discussions of the proofs of some formulations of the implicit function theorem. The early history of the theorem is recalled in Chapter two, highlighting the pioneering work of Newton and Lagrange and culminating with the first rigorous formulation by Cauchy for the case of holomorphic functions.

Chapter three is entirely devoted to the discussion and proof of the classical formulation of the theorem for Euclidean space. Actually it has two (or three) proofs. The first one is the classical proof one finds in an elementary calculus course, using no more than the Taylor expansion and the mean-value theorem. The second proof uses the contraction-mapping fixed-point theorem valid in any complete metric space. This chapter also includes a proof of the equivalence of the implicit and inverse function theorems, and a formulation and proof of the rank theorem for Euclidean spaces.

The chapter ends with an example that shows that the hypothesis requiring the derivative to be continuous cannot be omitted.

Chapter four starts by showing how the implicit function theorem can be used to prove the existence of solutions to ordinary differential equations. Next it has an application to numerical methods to solve systems of nonlinear equations. The last two sections of this chapter are devoted to the formulation and proof of the implicit function theorem in geometric analysis preparing the way for the most sophisticated applications and formulations in subsequent chapters.

Chapter five is devoted to other variations of the implicit function theorem, either for holomorphic maps or for functions with a degenerate Jacobian matrix.

In the last chapter, the authors treat advanced versions of the implicit function theorem, culminating with a complete proof of the celebrated Nash-Moser implicit function theorem.

The authors have taken some care to make the book self-contained, and as such a well-motivated undergraduate student can profitably read many parts of it, and the whole book is within the reach of a first-year graduate student. As the authors point out in the introduction, the best way to think about the implicit function theorem is like a way of looking at a problem or problems. An Ansatz, as the physicist like to say. Therefore the validity or falsity of this theorem in other settings is rather important. However, given that the focus of the book is wholly analytic, the reader may wonder if this particular theorem has not appeared in other parts of the geometric tree. Perhaps I may be allowed to point out that in algebraic geometry it is a rather elementary fact that for the Zariski topology the implicit function theorem is false: There are maps, the so-called étale morphisms, between algebraic varieties (or schemes) that induce isomorphisms on the corresponding tangent spaces but may not be local isomorphisms: the Zariski topology is too coarse. The correct formulation of the implicit function theorem in this context is the fact that an étale map induces isomorphisms on the completions of the corresponding local rings, under adequate conditions. Somehow, this may be seen as the corresponding properties of a local isomorphism. This remark may be far from the goals of the book under review, but it shows how this way of looking at some problems, this Ansatz, also blossoms in the fertile ground of algebraic geometry.

Felipe Zaldivar is Professor of Mathematics at the Universidad Autonoma Metropolitana-I, in Mexico City. His e-mail address is fz@xanum.uam.mx.