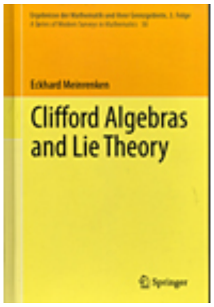
Clifford Algebras and Lie Theory



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MAA REVIEW | TABLE OF CONTENTS

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Around 1913, Elie Cartan was working on the classification of irreducible representations of simple Lie algebras. For the case of the Lie algebra of the orthogonal group, he found a rather special representation that could not be accounted for, as the other ones, by the usual constructions involving tensors. He called these representations *spin representations*, and the elements of the corresponding vector space he called *spinors*. Brauer and Weyl, in 1935, were the first to systematize these spin representations in terms of the language of Clifford algebras, but it was Chevalley in his classic monograph, *The Algebraic Theory of Spinors and Clifford Algebras* (Columbia University Press, 1954, reprinted by Springer in 1996 as Vol. 2 of his Collected Works), who gave a complete an self-contained exposition of the theory valid over any field. It was a classic, clear, and totally algebraic approach that at the same time clarified the relation between the exterior and Clifford algebras defined by a vector space with a symmetric bilinear form.

Nowadays one can find expositions of Clifford algebras ranging from introductory textbooks like D. J. H. Garling's Clifford Algebras: An Introduction (Cambridge, 2011), to monographs with applications in Physics like R. Penrose and W. Rindler's *Spinors and Space-Time* (2 Vols. Cambridge, 1985-1986), but Chevalley's monograph remains the best of all: concise, complete and crystal clear.

The book under review starts by recalling the basics of the theory of (symmetric) bilinear and quadratic forms on finite dimension vector spaces over an arbitrary field of characteristic different from 2, up to the Cartan-Dieudonné theorem (any orthogonal transformation can be written as composition of reflections) and Witt's theorem (any isometry between subspaces of a quadratic vector space V can be extended to an orthogonal transformation of V). With these foundations, and assuming the basic notions of Lie groups, Chapter one moves on to discuss the structure of the orthogonal groups and their Lie algebras, when the base field are the real or complex numbers. The last section of this chapter discusses the case of a symplectic bilinear form and the corresponding isotropic subspaces, with particular emphasis on the space of maximal isotropic subspaces of the given vector space, the Lagrangian-Grassmannian manifold. The highlights are theorems characterizing the Lagrangian-Grassmannian as a homogeneous space, in the real or complex case.

Chapter two starts collecting the construction and main properties of the exterior algebra associated to any vector space, including its universal property, functoriality, its derivations and duality pairings. Following Chevalley, the Clifford algebra associated to a vector space with a symmetric bilinear form is defined as a generalization of the exterior algebra taking into account the presence of the bilinear form. After proving the basic properties of Clifford algebras, discussing several examples (including the classical low dimensional real cases, e.g., the Hamiltonian quaternions), the author obtains the isomorphism from the Clifford algebra Cl(V,B) to the endomorphism algebra of the exterior product of the vector space V. The inverse of this isomorphism, the quantization map, allows us to interpret the Clifford algebra Cl(V,B) as the exterior product of V with a deformed associative product, and the last section of Chapter two is devoted to explain, in terms of differential operators and graded Poisson structures, why it is indeed a quantization.

Chapters three and four are devoted to spinors and the spin representation using the Clifford algebra of a vector space with a split bilinear form. Chapter three includes a detailed study of pure spinors and their relation to the Lagrangian subspaces, and an exposition, following Chevalley, of Cartan's triality principle.

Chapter five is devoted to enveloping algebras. The main results include the Poincaré-Birkhoff-Witt theorem (the natural quantization map from the symmetric algebra of a given Lie algebra to the enveloping algebra is an isomorphism) whose proof, following Petracci, is somehow similar to the proof in Chapter two, that the quantization map for Clifford algebras is an isomorphism.

Chapters six and seven treat the classical and quantum Weil algebras. The exposition follows the path traced in previous chapters. Thus, the Weil algebra is seem as a differential algebra given by the tensor product of the symmetric and exterior algebras of the dual of the given Lie algebra. And their main properties are proven in chapter six. Next, since the Clifford algebra and the enveloping algebras can be seen as quantizations of the exterior algebra and the symmetric algebra, respectively, the quantum Weil algebra is defined as the tensor product of these algebras in close analogy with the classical Weil algebra. The main result of chapter seven is that the quantization map from the Weil algebra to the quantum Weil algebra is an isomorphism of differential algebras. A consequence is Duflo's isomorphism for the case of quadratic Lie algebras.

In chapter eight the author applies the results of the previous chapters to complex reductive Lie algebras, proving several classical results in this context, from the "strange formula" of Freudenthal-de Vries to the theory of multiplets of representations for equal rank Lie subalgebras and its interpretation in terms of the cubic Dirac operator. More details on the geometric Dirac operator are discussed in chapter nine.

After reviewing some facts from the homology and cohomology of Lie algebras, in chapter ten we find the classical Hopf-Koszul-Samelson theorem, which identifies the invariants, of the Lie algebra action on the exterior algebra of the underlying vector space, with the exterior algebra of the subspace of primitive elements. With these preliminaries, we reach the important final chapter, devoted to prove several recent results, starting with the Clifford algebra analogue of the Hopf-Koszul-Samelson theorem of B. Kostant and many other results viewed now as quantizations.

As usual with the Ergebnisse series, the monograph under review provides the interested reader an introduction to an important topic, putting in context several important recent developments dispersed in the specialized literature. The book is addressed to mathematicians, with some remarks when the topic has some relevance to mathematical physics, as can be gathered from the use of the Dirac operators.

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