DEPARTAMENTO DE MATEMÁTICAS

TITULO:

MULTIPLIERS IN PERFECT LOCALLY $m$-CONVEX ALGEBRAS

AUTORES:

MARINA HARALAMPIDOU, LOURDES PALACIOS AND CARLOS SIGNORET
MULTIPLIERS IN PERFECT LOCALLY $m$-CONVEX ALGEBRAS

MARINA HARALAMPIDOU, LOURDES PALACIOS AND CARLOS SIGNORET

ABSTRACT. In this paper we describe the multiplier algebra of a perfect complete locally $m$-convex algebra with an approximate identity and with complete Arens-Michael normed factors.

1. INTRODUCTION AND PRELIMINARIES

Multipliers are important in various areas of mathematics where an algebra structure appears (see e.g. [1]; for (non-normed) topological algebras cf. e.g. [1]).

The algebras considered throughout are taken over the field of complexes C. Denote by $L(E)$ the algebra of all linear operators on an algebra $E$.

**Definition 1.1.** A mapping $T : E \to E$ is called a left (right) multiplier on $E$ if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in E$; it is called a two-sided multiplier on $E$ if it is both a left and a right multiplier.

It is known that if $E$ is a proper algebra, namely $xE = \{\theta\}$ implies $x = \theta$ or $Ex = \{\theta\}$ implies $x = \theta$, where $\theta$ denotes the null element of $E$, then any two-sided multiplier on $E$ is automatically a linear mapping [6, p. 20].

In the sequel, a two-sided multiplier will be called in short, a multiplier. Let us denote by $M_l(E)$ the set of all left multipliers on $E$, by $M_r(E)$ the set of all right multipliers on $E$ and by $M(E)$ that of all multipliers on $E$. Note that, by definition, $M(E) = M_l(E) \cap M_r(E)$.

Obviously $M(E)$ is a subalgebra of $L(E)$ in case the algebra is proper. The same holds for $M_l(E)$ and $M_r(E)$. Now, for $x \in E$, the operator $l_x$ on $E$ given by $l_x(y) = xy, y \in E$, is, due to the associativity of $E$, a left multiplier. Similarly, we can also define the right multiplier with respect to $x \in E$, say $r_x$.

It is known that if $E$ is a proper algebra, then the mapping

$$F : E \to M_l(E) \text{ given by } x \mapsto l_x$$

defines an algebra monomorphism which identifies $E$ with a subalgebra of $M_l(E)$. Moreover, $E$ is a left ideal of the algebra $M_l(E)$ (see [4, p. 1933, Proposition 2.2]). A similar result is also valid for right multipliers. For multipliers, the algebra $E$ can be identified with a two-sided ideal in $M(E)$ (ibid, p. 1934, Corollary 2.3).

1991 Mathematics Subject Classification. Primary 46H05; Secondary 46H10, 46K05.

Key words and phrases. Proper algebra, Arens-Michael decomposition, multiplier algebra, perfect projective system, perfect locally $m$-convex algebra.
MULTIPLIERS IN PERFECT LOCALLY $m$-CONVEX ALGEBRAS

MARINA HARALAMPIDOU, LOURDES PALACIOS AND CARLOS SIGNORET

Abstract. In this paper we describe the multiplier algebra of a perfect complete locally $m$-convex algebra with an approximate identity and with complete Arens-Michael normed factors.

1. Introduction and Preliminaries

Multipliers are important in various areas of mathematics where an algebra structure appears (see e.g. [1]; for (non-normed) topological algebras cf. e.g. [4]).

The algebras considered throughout are taken over the field of complexes $\mathbb{C}$. Denote by $L(E)$ the algebra of all linear operators on an algebra $E$.

Definition 1.1. A mapping $T : E \rightarrow E$ is called a left (right) multiplier on $E$ if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in E$; it is called a two-sided multiplier on $E$ if it is both a left and a right multiplier.

It is known that if $E$ is a proper algebra, namely $xE = \{\theta\}$ implies $x = \theta$ or $Ex = \{\theta\}$ implies $x = \theta$, where $\theta$ denotes the null element of $E$, then any two-sided multiplier on $E$ is automatically a linear mapping [6, p. 20].

In the sequel, a two-sided multiplier will be called in short, a multiplier. Let us denote by $M_l(E)$ the set of all left multipliers on $E$, by $M_r(E)$ the set of all right multipliers on $E$ and by $M(E)$ that of all multipliers on $E$. Note that, by definition, $M(E) = M_l(E) \cap M_r(E)$.

Obviously $M(E)$ is a subalgebra of $L(E)$ in case the algebra is proper. The same holds for $M_r(E)$ and $M_l(E)$. Now, for $x \in E$, the operator $l_x$ on $E$ given by $l_x(y) = xy, y \in E$, is, due to the associativity of $E$, a left multiplier. Similarly, we can also define the right multiplier with respect to $x \in E$, say $r_x$.

It is known that if $E$ is a proper algebra, then the mapping

$$F : E \rightarrow M_l(E) \text{ given by } x \mapsto l_x$$

defines an algebra monomorphism which identifies $E$ with a subalgebra of $M_l(E)$. Moreover, $E$ is a left ideal of the algebra $M_l(E)$ (see [3, p. 1933, Proposition 2.2]). A similar result is also valid for right multipliers. For multipliers, the algebra $E$ can be identified with a two-sided ideal in $M(E)$ (ibid, p. 1934, Corollary 2.3).

1991 Mathematics Subject Classification. Primary 46H05; Secondary 46H10, 46K05.

Key words and phrases. Proper algebra, Arens-Michael decomposition, multiplier algebra, perfect projective system, perfect locally $m$-convex algebra.
Definition 1.2. An approximate identity in a topological algebra $E$ is a net $(e_\delta)_{\delta \in \Delta}$ such that for each $x \in E$ we have:

$$(x - xe_\delta) \to 0 \quad \text{and} \quad (x - e_\delta x) \to 0$$

for all $x \in E$.

Note that an algebra with an approximate identity is proper. In this paper we describe the multiplier algebra $M(E)$ in the case where $E$ is a certain complete locally $m$-convex algebra with an approximate identity.

For the sake of completeness, we recall what we mean by the "Arens-Michael decomposition" ([7, p. 88, Theorem 3.1]).

Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete locally $m$-convex algebra and

$$\rho_\alpha : E \to E/\ker(p_\alpha) \cong E_\alpha$$

the respective quotient maps. Then $\rho_\alpha(p_\alpha(x)) = x + \ker(p_\alpha)$ defines an algebra norm, so that $E_\alpha$ is a normed algebra and the morphisms $\rho_\alpha$, $\alpha \in \Lambda$ are continuous.

$E_\alpha$, $\alpha \in \Lambda$ denotes the completion of $E_\alpha$ (with respect to $\rho_\alpha$). $\Lambda$ is endowed with a partial order by putting $\alpha \leq \beta$ if and only if $p_\alpha(x) \leq p_\beta(x)$ for every $x \in E$. Thus, $\ker(p_\beta) \subseteq \ker(p_\alpha)$ and hence the continuous (onto) morphism $f_{\alpha\beta} : E_\beta \to E_\alpha : \beta \mapsto f_{\alpha\beta}(x_\beta) = x_\alpha$, $\alpha \leq \beta$ is defined. Moreover, $f_{\alpha\beta}$ is extended to a continuous morphism $\tilde{f}_{\alpha\beta} : \tilde{E}_\beta \to \tilde{E}_\alpha$, $\alpha \leq \beta$. Thus, $(E_\alpha, f_{\alpha\beta})$, $(\tilde{E}_\alpha, \tilde{f}_{\alpha\beta})$, $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$ are projective systems of normed (resp. Banach) algebras, so that $E \cong \lim E_\alpha \cong \lim \tilde{E}_\alpha$ (Arens-Michael decomposition) within topological algebra isomorphism.

In [3, p. 1934, Theorem 3.1], it is shown that, in a special case, the algebra $M(E)$ is a subalgebra of $L(E)$, the algebra of all continuous linear operators in $E$; for completeness, we refer it here.

Theorem 1.3. Let $(E, (p_\alpha)_{\alpha \in \Lambda})$ be a complete locally $m$-convex algebra with an approximate identity $(e_\delta)_{\delta \in \Delta}$. Suppose that each factor $E_\alpha = E/\ker(p_\alpha)$ in the Arens-Michael decomposition of $E$ is complete. Then each multiplier $T$ of $E$ is continuous, viz. $M(E)$ is a subalgebra of $L(E)$.

2. Perfectness and Multipliers in locally $m$-convex Algebras

To proceed, we use the notion of a perfect projective system as it appeared in [2, p. 199, Definition 2.7]. To fix notation, we repeat it.

Definition 2.1. A projective system $\{(E_\alpha, f_{\alpha\beta})\}_{\alpha \in \Lambda}$ of topological algebras is called perfect, if the restrictions to the projective limit algebra

$$E = \lim E_\alpha = \{(x_\alpha) \in \prod_{\alpha \in \Lambda} E_\alpha : f_{\alpha\beta}(x_\beta) = x_\alpha, \text{ if } \alpha \leq \beta \in \Lambda\}$$

of the canonical projections $\pi_\alpha : \prod_{\alpha \in \Lambda} E_\alpha \to E_\alpha$, $\alpha \in \Lambda$, namely, the (continuous algebra) morphisms

$$f_\alpha = \pi_\alpha \circ \lim E_\alpha : E \to E_\alpha, \quad \alpha \in \Lambda,$$

are onto maps. The resulting projective limit algebra $E = \lim E_\alpha$ is then called a perfect (topological) algebra.
Definition 2.2. In the sequel, by the term perfect locally \( m \)-convex algebra we mean a locally \( m \)-convex algebra \((E, (p_\alpha)_{\alpha \in \Lambda})\) for which the respective Arens-Michael projective system \(\{(E_\alpha, f_{\alpha \beta})\}_{\alpha \in \Lambda}\) is perfect.

Every Fréchet locally \( m \)-convex algebra \((E, (p_\alpha)_{\alpha \in \mathbb{N}})\) gives a perfect projective system of normed algebras, and thus it is a perfect algebra (see [2], and [5]).

Example 2.3. Let \(E\) be a non-complete normed algebra. Take \(E = E_\alpha\) for each \(\alpha \in \Lambda\) and, for \(\alpha \leq \beta\), let \(f_{\alpha \beta} : E_\beta \to E_\alpha\) be the identity map. Then \(\Delta = \lim_{\alpha \to \beta} E_\alpha\), the diagonal algebra, is a perfect locally \( m \)-convex algebra, but \(\Delta\) is non-complete.

Let \(E = (E, (p_\alpha)_{\alpha \in \Lambda})\) be a complete perfect locally \( m \)-convex algebra with an approximate identity and such that each factor \(E_\alpha\) of its Arens-Michael decomposition \(\{E_\alpha, f_{\alpha \beta}, \alpha \leq \beta\}\) is complete.

Remark 2.4. If \(\phi\) is the isomorphism \(E \to \lim E_\alpha\) given by \(\phi(x) = (x_\alpha)_{\alpha \in \Lambda}\); then, for each \(\alpha \in \Lambda\), \(\rho_\alpha = f_\alpha \circ \phi\). Therefore, \(\ker p_\alpha = \ker (f_\alpha \circ \phi)\).

Remark 2.5. By the hypothesis of perfection, each \(f_\beta\) is surjective, so each time we have an element \(x_\beta \in E_\beta\), we can choose an element \(\omega \in E\) such that \(\omega_\beta = x_\beta\), and consequently \(\omega_\alpha = f_{\alpha \beta}(x_\beta) = x_\alpha\), whenever \(\alpha \leq \beta\).

For each \(\alpha \leq \beta\), we define the map \(h_{\alpha \beta} : M(E_\beta) \to M(E_\alpha)\) given by
\[
[h_{\alpha \beta}(T_\beta)](x_\alpha) = f_{\alpha \beta}(T_\beta(x_\beta))
\]
which is well defined, according to the following lemma.

Lemma 2.6. Let \((E, (p_\alpha)_{\alpha \in \Lambda})\) be a complete perfect locally \( m \)-convex algebra with an approximate identity \((e_\delta)_{\delta \in \Delta}\) and such that each factor \(E_\alpha\) of its Arens-Michael decomposition is complete. Then \(\ker f_{\alpha \beta}\) is \(T_\beta\)-invariant for each \(T_\beta \in M(E_\beta)\), that is, \(T_\beta(\ker f_{\alpha \beta}) \subseteq \ker f_{\alpha \beta}\), if \(\alpha \leq \beta\), and the map \(h_{\alpha \beta}\) is a well-defined continuous multiplicative linear mapping.

Proof. Take \(x_\beta \in \ker f_{\alpha \beta}\). Since \(E\) has an approximate identity \((e_\delta)_{\delta \in \Delta}\) and multipliers over Banach algebras are continuous (see [6, p. 20, Theorem 1.1.1]), then
\[
f_{\alpha \beta}(T_\beta(x_\beta)) = f_{\alpha \beta}(T_\beta(\lim \delta e_\delta)) = f_{\alpha \beta}(\lim \delta T_\beta(x_\beta e_\delta)) = \lim \delta f_{\alpha \beta}(T_\beta(x_\beta e_\delta)) = \lim \delta [f_{\alpha \beta}(x_\beta e_\delta)] = 0.
\]

We claim that \(h_{\alpha \beta}(T_\beta)\) is well-defined. For that, let \(\alpha \leq \beta\), \(x \in E\) such that \(x_\alpha = x'_\alpha\) and \(T_\beta \in M(E_\beta)\); then \(0 = x_\alpha - x'_\alpha = \rho_\alpha(x) - \rho_\alpha(x') = \rho_\alpha(x - x')\) and hence \(0 = (f_\alpha \circ \phi)(x - x') = (f_\alpha \circ f_\beta \circ \phi)(x - x')\), which implies that \((f_\beta \circ \phi)(x - x') \in \ker f_{\alpha \beta}\). Since \(\ker f_{\alpha \beta}\) is \(T_\beta\)-invariant, \(T_\beta((f_\beta \circ \phi)(x - x')) \in \ker f_{\alpha \beta}\), too, and therefore
\[
0 = f_{\alpha \beta}(T_\beta((f_\beta \circ \phi)(x - x'))) = f_{\alpha \beta}(T_\beta(\rho_\beta(x - x')) = f_{\alpha \beta}(T_\beta(x_\beta - x'_\beta)) = f_{\alpha \beta}(T_\beta(x_\beta)) - f_{\alpha \beta}(T_\beta(x'_\beta)),
\]
that is, \(f_{\alpha \beta}(T_\beta(x_\beta)) = f_{\alpha \beta}(T_\beta(x'_\beta))\). This proves the claim.

Moreover, \(h_{\alpha \beta}(T_\beta)\) is actually a multiplier on \(E_\alpha\). For, let \(x_\alpha\) and \(y_\alpha\) be two elements in \(E_\alpha\). Then
\[ [h_{\alpha\beta}(T_{\beta})](x_{\alpha}y_{\alpha}) = f_{\alpha\beta}(T_{\beta}(x_{\beta}y_{\beta})) = f_{\alpha\beta}(x_{\beta}T_{\beta}(y_{\beta})) = f_{\alpha\beta}(x_{\beta})f_{\alpha\beta}(T_{\beta}(y_{\beta})) = x_{\alpha}(f_{\alpha\beta}(T_{\beta}(y_{\beta})) = x_{\alpha}(h_{\alpha\beta}(T_{\beta})(y_{\alpha}) \]

and so, \( h_{\alpha\beta}(T_{\beta}) \) is a right multiplier. In a similar way, one can prove that \( h_{\alpha\beta}(T_{\beta}) \) is a left multiplier.

It is easily seen that \( h_{\alpha\beta} \) is a linear mapping. Moreover, \( h_{\alpha\beta} \) is multiplicative. For that, take \( T_{\beta}, S_{\beta} \in M(E_{\beta}) \). We have

\[
[h_{\alpha\beta}(T_{\beta} \circ S_{\beta})](x_{\alpha}) = f_{\alpha\beta}((T_{\beta} \circ S_{\beta})(x_{\beta})) = f_{\alpha\beta}(T_{\beta}(S_{\beta}(x_{\beta}))). \tag{2.1}
\]

On the other hand, since the system is perfect, we can choose \( \omega \in E \) (equivalently \( (\omega_{\alpha})_{\alpha \in \Lambda} \)) such that \( f_{\alpha\beta}(S_{\beta}(x_{\beta})) = \omega_{\alpha} \); note that \( f_{\alpha\beta}(\omega_{\beta}) = \omega_{\alpha} \) too. Then \( S_{\beta}(x_{\beta}) = \omega_{\beta} \in \ker f_{\alpha\beta} \). But, since \( \ker f_{\alpha\beta} \) is \( T_{\beta} \)-invariant, we have \( T_{\beta}(S_{\beta}(x_{\beta}) - \omega_{\beta}) \in \ker f_{\alpha\beta} \), and thus \( f_{\alpha\beta}(T_{\beta}(S_{\beta}(x_{\beta})) = f_{\alpha\beta}(T_{\beta}(\omega_{\beta})) \). Besides,

\[
f_{\alpha\beta}(T_{\beta}(S_{\beta}(x_{\beta}))) = f_{\alpha\beta}(T_{\beta}(\omega_{\beta})) = h_{\alpha\beta}(T_{\beta})(\omega_{\alpha}) = h_{\alpha\beta}(T_{\beta})(f_{\alpha\beta}(S_{\beta}(x_{\beta}))) = h_{\alpha\beta}(T_{\beta})(h_{\alpha\beta}(S_{\beta})(x_{\alpha})) = h_{\alpha\beta}(T_{\beta} \circ h_{\alpha\beta}(S_{\beta}))(x_{\alpha}).
\]

The last, in connection with (2.1) gives the multiplicativity of \( h_{\alpha\beta} \).

Next, we prove that \( h_{\alpha\beta} \) is continuous. Since \( f_{\alpha\beta} : E_{\beta} \to E_{\alpha} \) is a continuous mapping between normed algebras, there exists a constant \( K > 0 \) such that \( p_{\alpha}(f_{\alpha\beta}(y_{\beta})) \leq K \cdot p_{\beta}(y_{\beta}) \) for each \( y_{\beta} \in E_{\beta} \). In particular,

\[
p_{\alpha}(f_{\alpha\beta}(T_{\beta}(x_{\beta}))) \leq K \cdot p_{\beta}(T_{\beta}(x_{\beta})) \text{ for each } x_{\beta} \in E_{\beta}. \tag{2.2}
\]

Taking the supremum on the right hand of (2.2) and since \( M(E_{\beta}) \) is a Banach algebra (see [6, p. 20, Theorem 1.1.1]), we get

\[
p_{\alpha}(f_{\alpha\beta}(T_{\beta}(x_{\beta}))) \leq K \cdot p_{\beta}(T_{\beta}(x_{\beta})) \leq K \sup_{p_{\beta}(x_{\beta}) \leq 1} \{ p_{\beta}(T_{\beta}(x_{\beta})) \} \leq K \| T_{\beta} \|_{\beta} \tag{2.3}
\]

for every \( x_{\beta} \in E_{\beta} \) with \( p_{\beta}(x_{\beta}) \leq 1 \), and where \( \| \|_{\beta} \) is the norm in the multiplier algebra \( M(E_{\beta}) \). Since \( f_{\alpha\beta}(T_{\beta}(x_{\beta})) = [h_{\alpha\beta}(T_{\beta})](x_{\alpha}) \) whenever \( \alpha \leq \beta \) (hence \( p_{\alpha}(x_{\alpha}) \leq p_{\beta}(x_{\beta}) \)), then \( p_{\alpha}([h_{\alpha\beta}(T_{\beta})](x_{\alpha})) \leq K \| T_{\beta} \|_{\beta} \) for every \( x_{\alpha} \in E_{\alpha} \) with \( p_{\alpha}(x_{\alpha}) \leq 1 \) by (2.3). Taking now the supremum in this latter relation, we have

\[
\sup_{p_{\alpha}(x_{\alpha}) \leq 1} p_{\alpha}([h_{\alpha\beta}(T_{\beta})](x_{\alpha})) \leq K \| T_{\beta} \|_{\beta}. \text{ Thus } \| h_{\alpha\beta}(T_{\beta}) \|_{\alpha} \leq K \| T_{\beta} \|_{\beta}, \text{ namely, each } h_{\alpha\beta} \text{ is continuous.}
\]

So far, we have the family of topological algebras \( M(E_{\alpha}) \) and the family of multiplicative continuous linear mappings \( h_{\alpha\beta} : M(E_{\beta}) \to M(E_{\alpha}), \alpha \leq \beta \in \Lambda \). Actually, they form a projective system. In fact, if \( \alpha \leq \beta \leq \gamma \), then \( f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma} \), and therefore

\[
[h_{\alpha\gamma}(T_{\gamma})](x_{\alpha}) = f_{\alpha\gamma}(T_{\gamma}(x_{\gamma})) = (f_{\alpha\beta} \circ f_{\beta\gamma})(T_{\gamma}(x_{\gamma})) = f_{\alpha\beta}(f_{\beta\gamma}(T_{\gamma}(x_{\gamma})) = f_{\alpha\beta}([h_{\beta\gamma}(T_{\gamma})](x_{\beta})) = [h_{\alpha\beta}(h_{\beta\gamma}(T_{\gamma}))](x_{\alpha}) = [h_{\alpha\beta} \circ h_{\beta\gamma}](T_{\gamma})(x_{\alpha})
\]

for each \( x_{\alpha} \in E_{\alpha} \). That is, \( h_{\alpha\gamma}(T_{\gamma}) = (h_{\alpha\beta} \circ h_{\beta\gamma})(T_{\gamma}) \) for each \( T_{\gamma} \in M(E_{\gamma}) \), which implies that \( h_{\alpha\gamma} = h_{\alpha\beta} \circ h_{\beta\gamma}; \text{ it is clear that } h_{\alpha\alpha} = Id_{M(E_{\alpha})}. \)
Thus, we have the projective system of Banach algebras \( \{(M(E_\alpha), h_{\alpha\beta})\}_{\alpha\in\Lambda} \) and we can take its inverse limit, \( \lim M(E_\alpha) \).

Now, we prove a lemma extracted from the proof of Theorem 3.1 in [3, p. 1934] that will be useful in the sequel.

**Lemma 2.7.** Let \((E, (p_\alpha)_{\alpha\in\Lambda})\) be a locally m-convex algebra with an approximate identity \((e_\delta)_{\delta\in\Delta}\) and let \(T \in M(E)\). Then, for each \(\alpha \in \Lambda\), \(p_\alpha\) is \(T\)-invariant; that is, \(T(\ker p_\alpha) \subseteq \ker p_\alpha\).

**Proof.** Take \(x \in \ker p_\alpha\). Since the seminorms are continuous, for \(\varepsilon > 0\), there exists an index \(\delta_0 \in \Delta\) such that \(p_\alpha(T(x) - T(x)e_\delta) < \varepsilon\) whenever \(\delta \geq \delta_0\). We have

\[
p_\alpha(T(x)) = p_\alpha(T(x - xe_\delta + xe_\delta)) = p_\alpha(T(x) - T(xe_\delta + xe_\delta)) \leq p_\alpha(T(x) - T(xe_\delta)) + p_\alpha(T(xe_\delta)) = p_\alpha(T(x) - T(x)e_\delta + p_\alpha(xT(e_\delta)) \leq p_\alpha(T(x) - T(x)e_\delta) + p_\alpha(xT(e_\delta)) < \varepsilon.
\]

Since this is true for an arbitrary \(\varepsilon > 0\), we conclude that \(p_\alpha(T(x)) = 0\), that is, \(T(x) \in \ker p_\alpha\). \(\Box\)

Now we state our main Theorem.

**Theorem 2.8.** Let \((E, (p_\alpha)_{\alpha\in\Lambda})\) be a complete locally m-convex algebra with an approximate identity \((e_\delta)_{\delta\in\Delta}\), such that the respective projective system is perfect and each factor \(E_\alpha = E/\ker p_\alpha\) in its Arens-Michael decomposition is complete. Then \(M(E) \cong \lim M(E_\alpha)\) within a topological algebra isomorphism.

**Proof.** Take \(T \in M(E)\). Due to Lemma 2.7, \(T\) induces a well-defined map \(T_\alpha : E_\alpha \to E_\alpha\) such that \(T_\alpha \circ p_\alpha = p_\alpha \circ T\) for each \(\alpha \in \Lambda\), that is,

\[
T_\alpha(x_\alpha) = T_\alpha(p_\alpha(x)) = p_\alpha(T(x)) = T(x)_\alpha\text{ for each }x \in E.
\]

Since for \(x_a, y_a \in E_\alpha\),

\[
T_\alpha(x_a y_a) = T_\alpha(T(x y)) = p_\alpha(x T(y)) = x_a T(y)_\alpha = x_a T_\alpha(y)_\alpha,
\]

\(T_\alpha\) is a right multiplier. In a similar way it can be shown that it is a left multiplier, as well.

Note also that \((T_\alpha)_{\alpha\in\Lambda}\) is an element of \(\lim M(E_\alpha)\). Indeed, for \(\alpha \leq \beta\) and \(p_\alpha(x) = x_\alpha \in E_\alpha\), we have

\[
[h_{\alpha\beta}(T_\beta)](\rho_\alpha(x)) = [h_{\alpha\beta}(T_\beta)](x_\alpha) = f_{\alpha\beta}(T_\beta((x_\beta)) = f_{\alpha\beta}(T_\beta(p_\beta(x))) = f_{\alpha\beta}(T_\beta(T(x))) = f_{\alpha\beta}(f_{\beta\alpha} \circ \phi)(T(x)) = f_{\alpha\beta}(f_{\beta\alpha} \circ \phi)(T(x)) = (f_\alpha \circ \phi)(T(x)) = \rho_\alpha(T(x)) = T_\alpha(p_\alpha(x)).
\]

Therefore \(h_{\alpha\beta}(T_\beta) = T_\alpha\) if \(\alpha \leq \beta\).

Now we define the map \(\Phi : M(E) \to \lim M(E_\alpha)\) by \(\Phi(T) = (T_\alpha)_{\alpha\in\Lambda}\), which obviously is linear. Moreover, for \(T, S \in M(E)\) and \(x_\alpha \in E_\alpha\), we have

\[
p_\alpha(\Phi(T \circ S)) = (T \circ S)_\alpha(x_\alpha) = (T \circ S)(x)_\alpha = (T(S(x))_\alpha = T_\alpha(S(x)_\alpha = T_\alpha(S_\alpha(x_\alpha)) = (T_\alpha \circ S_\alpha)(x_\alpha),
\]

which implies that \((T \circ S)_\alpha = T_\alpha \circ S_\alpha\), and therefore \(\Phi(T \circ S) = \Phi(T) \circ \Phi(S)\), namely, \(\Phi\) is multiplicative.

Next, we show that \(\Phi\) is one to one. For that, take \(T, S \in M(E)\) such that \((T_\alpha)_{\alpha\in\Lambda} = \Phi(T) = \Phi(S) = (S_\alpha)_{\alpha\in\Lambda}\); then \(T_\alpha = S_\alpha\) for each \(\alpha \in \Lambda\). Therefore
\( \rho_\alpha \circ T = \rho_\alpha \circ S \) for each \( \alpha \in \Lambda \); then \( T = S \). Moreover, \( \Phi \) is an onto map. Indeed, for \( (W_\alpha)_{\alpha \in \Lambda} \in \varprojlim M(E_\alpha) \) define the map
\[
W : E \to E \text{ by } W(x) = \phi^{-1}((W_\alpha(x_\alpha))_{\alpha \in \Lambda}),
\]
which obviously is linear. Also
\[
W(xy) = \phi^{-1}((W_\alpha(xy)_\alpha)_{\alpha \in \Lambda}) = \phi^{-1}((W_\alpha(x_\alpha y_\alpha))_{\alpha \in \Lambda}) = \phi^{-1}((x_\alpha W_\alpha(y_\alpha))_{\alpha \in \Lambda}) = \phi^{-1}((x_\alpha)_\alpha\phi^{-1}((W_\alpha(y_\alpha))_{\alpha \in \Lambda}) = xW(y),
\]
and similarly on the other side, so \( W \) is a multiplier on \( E \). Finally, it is clear that \( \Phi(W) = (W_\alpha)_{\alpha \in \Lambda} \).

We claim that \( \Phi \) is continuous. By [3, p. 1934, Theorem 3.1], \( M(E) \) is a subalgebra of \( \mathcal{L}(E) \), the algebra of all continuous linear operators on \( E \), so that the topology on \( M(E) \) is the operator topology. Let us denote by
\[
g_\alpha : M(E) \to M(E_\alpha)
\]
the map \( g_\alpha(T) = T_\alpha \), which, by Lemma 2.7, is well defined and obviously linear.

Let us denote by \( h_\alpha : \varprojlim M(E_\alpha) \to M(E_\alpha) \) the canonical continuous homomorphism from the inverse limit to one of its factors. Note that \( h_\alpha \circ \Phi = g_\alpha \) holds for each \( \alpha \in \Lambda \).

Since \( \Phi \) is continuous if and only if, for each \( \alpha \in \Lambda \), \( h_\alpha \circ \Phi \) is continuous (see [7, p. 89, the proof of Theorem 3.1]), we have to prove that \( g_\alpha \) is continuous (for each \( \alpha \in \Lambda \)). Let us recall that the topology of \( M(E) \) can be given by the set of seminorms \((\bar{p}_\alpha, \alpha \in \Lambda)\) defined as
\[
\bar{p}_\alpha(T) = \sup_{p_\alpha(x) \leq 1} p_\alpha(T(x))
\]
for each \( T \in M(E) \). Further, the topology of \( M(E_\alpha) \) can be given by the norm \( \|S\|_\alpha = \hat{p}(S(x)) \) for each \( S \in M(E_\alpha) \), where, as usual, \( \hat{p} \) is the induced norm in \( E_\alpha \) given by \( \hat{p}(x_\alpha) = \hat{p}(x + \ker p_\alpha) = p_\alpha(x) \). The topology of \( \varprojlim M(E_\alpha) \) can be defined by the local base consisting of neighborhoods \( V = \bigcap_{i=1}^n h_\alpha^{-1}(V_\alpha) \), where \( V_\alpha \) is a basic neighborhood in \( M(E_\alpha) \).

Let \( \varepsilon_i > 0 \) be given and let \( V_\alpha = \{S \in M(E_\alpha) : \|S\|_\alpha < \varepsilon_i\} \) and \( U_\alpha = \{T \in M(E) : \bar{p}_\alpha(T) < \varepsilon_i\} \). We claim that
\[
T \in U_\alpha \iff T_\alpha \in V_\alpha.
\]
Indeed,
\[
T \in U_\alpha \iff \bar{p}_\alpha(T) < \varepsilon_i \iff \sup_{p_\alpha(x) \leq 1} p_\alpha(T(x)) < \varepsilon_i
\]
\[
\iff \sup_{\hat{p}(x_\alpha) \leq 1} \hat{p}(T(x_\alpha)) < \varepsilon_i \iff \max_{\hat{p}(x_\alpha) \leq 1} \hat{p}(T_\alpha(x_\alpha)) < \varepsilon_i \iff \|T_\alpha\|_\alpha < \varepsilon_i
\]
\[
\iff T_\alpha \in V_\alpha.
\]

Now, let \( V_\alpha \) be a basic neighborhood of 0 in \( M(E_\alpha) \), say
\[
V_\alpha = \{S \in M(E_\alpha) : \|S\|_\alpha < \varepsilon\}.
\]
Put \( U_\alpha = g_\alpha^{-1}(V_\alpha) \). Then \( U_\alpha = \{T \in M(E) : \bar{p}_\alpha(T) < \varepsilon\} \).

This implies the continuity of \( g_\alpha \) for each \( \alpha \in \Lambda \). Hence \( \Phi \) is continuous.

Finally, we show that \( \Phi \) is an open map. Let \( V = \bigcap_{i=1}^n h_\alpha^{-1}(V_\alpha) \) be a basic neighborhood of 0 in \( M(E) \). Take \( T \in V \); then \( T \in h_\alpha^{-1}(V_\alpha) \) for all \( i = 1, \ldots, n \). Therefore \( T_\alpha = h_\alpha(T) \in V_\alpha \) and, due to (2.4), \( T \in U_\alpha \). Then \( \Phi(T) \in U = (U_\alpha) \),
where $U_\alpha = U_{\alpha_i}$ for $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ and $U_\alpha = M(E_\alpha)$ otherwise. This proves that $\Phi$ is an open map, and the proof is complete. \hfill \Box

**Acknowledgments**

We want to thank Professor Mohamed Oudadess for the suggestion of Example 2.3 and for his fruitful comments.

We also want to thank the referees for their useful suggestions.

**REFERENCES**


1 Department of Mathematics, University of Athens, Panepistimioupolis, Athens 15784, Greece.

E-mail address: mharalam@math.uoa.gr;

2 Universidad Autónoma Metropolitana Iztapalapa, Mexico City 09340, Mexico;

E-mail address: pafa@xanum.uam.mx;

3 Universidad Autónoma Metropolitana Iztapalapa, Mexico City 09340, Mexico;

E-mail address: casi@xanum.uam.mx