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Vladimir V. Tkachuk

A *Cp*-Theory Problem Book

Special Features of Function Spaces

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Series Editors:
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Department of Mathematics
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Vladimir V. Tkachuk
Departamento de Matematicas
Universidad Autonoma Metropolitana-Iztapalapa
San Rafael Atlixco, Mexico City, Mexico

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Preface

This is the second volume of the series of books of problems in C_p-theory entitled *A C_p-Theory Problem Book*, i.e., this book is a continuation of the first volume subtitled *Topological and Function Spaces*. The series was conceived as an introduction to C_p-theory with the hope that each volume will also be used as a reference guide for specialists.

The first volume provides a self-contained introduction to general topology and C_p-theory and contains some highly nontrivial state-of-the-art results. For example, Sect. 1.4 presents Shapirovsky's theorem on the existence of a point-countable π-base in any compact space of countable tightness and Sect. 1.5 brings the reader to the frontier of the modern knowledge about realcompactness in the context of function spaces.

This present volume introduces quite a few topics from scratch but dealing with topology and C_p-theory is already a professional endeavour. The objective is to study the behaviour of general topological properties in function spaces and establish the results on duality of cardinal functions and classes with respect to the C_p-functor. The respective background includes a considerable amount of top-notch results both in topology and set theory; the author's obsession with keeping this work self-contained implied that an introduction to advanced set theory had to be provided in Sect. 1.1. The methods developed in this section made it possible to present a very difficult example of Todorčevič of a compact strong S-space.

Of course, it was impossible to omit the famous Baturov's theorem on coincidence of the Lindelöf number and extent in subspaces of $C_p(X)$ for any Lindelöf Σ-space X and the result of Christensen on σ-compactness of X provided that $C_p(X)$ is analytic. The self-containment policy of the author made it obligatory for him to give a thorough introduction to Lindelöf Σ-spaces in Sect. 1.3 and to the descriptive set theory in Sect. 1.4.

We use all topological methods developed in the first volume, so we refer to its problems and solutions when necessary. Of course, the author did his best to keep *every* solution as independent as possible, so a short argument could be repeated several times in different places.

The author wants to emphasize that if a postgraduate student mastered the material of the first volume, it will be more than sufficient to understand every problem and solution of this book. However, for a concrete topic, much less might be needed. Finally, let me outline some points which show the potential usefulness of the present work:

- *The only background needed is some knowledge of set theory and real numbers; any reasonable course in calculus covers everything needed to understand this book.*
- *The student can learn all of general topology required without recurring to any textbook or papers; the amount of general topology is strictly minimal and is presented in such a way that the student works with the spaces $C_p(X)$ from the very beginning.*
- *What is said in the previous paragraph is true as well if a mathematician working outside of topology (e.g., in functional analysis) wants to use results or methods of C_p-theory; he (or she) will find them easily in a concentrated form or with full proofs if there is such a need.*
- *The material we present here is up to date and brings the reader to the frontier of knowledge in a reasonable number of important areas of C_p-theory.*
- *This book seems to be the first self-contained introduction to C_p-theory. Although there is an excellent textbook written by Arhangel'skii (1992a), it heavily depends on the reader's good knowledge of general topology.*

Mexico City, Mexico Vladimir V. Tkachuk

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Introduction

The term "C_p-theory" was invented to abbreviate the phrase "The theory of function spaces endowed with the topology of pointwise convergence". The credit for the creation of C_p-theory must undoubtedly be given to Alexander Vladimirovich Arhangel'skii. The author is proud to say that Arhangel'skii also was the person who taught him general topology and directed his Ph.D. thesis. Arhangel'skii was the first to understand the need to unify and classify a bulk of heterogeneous results from topological algebra, functional analysis and general topology. He was the first to obtain crucial results that made this unification possible. He was also the first to formulate a critical mass of open problems which showed this theory's huge potential for development.

Later, many mathematicians worked hard to give C_p-theory the elegance and beauty it boasts nowadays. The author hopes that the work he presents for the reader's judgement will help to attract more people to this area of mathematics.

The main text of this volume consists of 500 statements formulated as problems; it constitutes Chap. 1. These statements provide a gradual development of many popular topics of C_p-theory to bring the reader to the frontier of the present-day knowledge. A complete solution is given to every problem of the main text.

The material of Chap. 1 is divided into five sections with 100 problems in each one. The sections start with an introductory part where the definitions and concepts to be used are given. The introductory part of any section *never exceeds two pages and covers everything that was not defined previously.* Whenever possible, we try to save the reader the effort of ploughing through various sections, chapters and volumes, so we give the relevant definitions in the current section not caring much about possible repetitions.

Chapter 1 ends with some bibliographical notes to give the most important references related to its results. The selection of references is made according to the author's preferences and by no means can be considered complete. However, a complete list of contributors to the material of Chap. 1 can be found in our bibliography of 300 items. It is my pleasant duty to acknowledge that I consulted the paper of Arhangel'skii (1998a) to include quite a few of its 375 references in my bibliography.

Sometimes, as we formulate a problem, we use without reference definitions and constructions introduced in other problems. The general rule is to try to find the relevant definition *not more than ten problems before.*

The first section of Chap. 1 deals with hereditary properties in $C_p(X)$. To understand the respective results, the reader needs a topological background including the ability to manage additional axioms of ZFC and apply strong and difficult methods of set theory. The pursuit of self-containment obliged the author to give an introduction to advanced set theory. In this section the reader can find the applications of continuum hypothesis, Martin's axiom, Jensen's axiom, Souslin trees and Luzin spaces.

The non-C_p material presented in Chap. 1 also includes an introduction to descriptive set theory and Lindelöf Σ-spaces. This helped to keep this work self-contained when we gave the proofs of Baturov's theorem on $C_p(X)$ for a Lindelöf Σ-space X and Christensen's theorem on σ-compactness of X provided that $C_p(X)$ is analytic. There are many topics in Chap. 1 which are developed up to the frontier of the present-day knowledge. In particular, Sect. 1.5 includes the famous Gerlits–Pytkeev theorem about coincidence of the Fréchet–Urysohn property and k-property in any space $C_p(X)$.

The complete solutions of all problems of Chap. 1 are given in Chap. 2. Chapter 3 begins with a selection of 100 statements which were proved as auxiliary facts in the solutions of the problems of the main text. This material is split into six sections to classify the respective results and make them easier to find. Chapter 4 consists of 100 open problems presented in ten sections with the same idea: to classify this bulk of problems and make the reader's work easier.

Chapter 4 also witnesses an essential difference between the organization of our text and the book of Arhangel'skii and Ponomarev (1974): *we never put unsolved problems in the main text as is done in their book.* All problems formulated in Chap. 1 are given complete solutions in Chap. 2 and the unsolved ones are presented in Chap. 4.

There is little to explain about how to use this book as a reference guide. In this case the methodology is not that important and the only thing the reader wants is to find the results he (or she) needs as fast as possible. To help with this, the titles of chapters and sections give the first approximation. To better see the material of a chapter, one can consult the second part of the Contents section where a detailed summary is given; it is supposed to cover all topics presented in each section. Besides, the index can also be used to find necessary material.

To sum up the main text, I believe that the coverage of C_p-theory will be reasonably complete and many of the topics can be used by postgraduate students who want to specialize in C_p-theory. Formally, this book can also be used as an introduction to general topology. However, it would be a somewhat biased introduction, because the emphasis is always given to C_p-spaces and the topics are only developed when they have some applications in C_p-theory.

To conclude, let me quote an old saying which states that the best way for one to learn a theorem is to prove it oneself. This text provides a possibility to do this. If the reader's wish is to read the proofs, then they are concentrated immediately after the main text.

Chapter 1
Duality Theorems and Properties of Function Spaces

This chapter presents some fundamental aspects of set theory, descriptive set theory, general topology and C_p-theory.

Section 1.1 introduces some advanced concepts of set theory. We give the statements and applications of the continuum hypothesis, Martin's axiom and Jensen's axiom. The next thing under the study is the behavior of spread, hereditary Lindelöf number and hereditary density in function spaces. The most important results of this section are the duality theorems for s^*, hd^* and hl^* (Problems 025–030) and Todorcevic's example of a strong S-space (Problem 098).

In Sect. 1.2 we deal with monolithity, stability and their generalizations. The principal results are presented as several generic theorems on duality between $\eta(\kappa)$-monolithity and $\theta(\kappa)$-stability, formulated in Problems 146–151.

Section 1.3 starts with Whyburn spaces and their properties. Next, we introduce Lindelöf Σ-spaces and their most important characterizations. The rest of the section is devoted to calibers, precalibers and small diagonals. The most important results include Baturov's theorem on extent in subspaces of $C_p(X)$ for a Lindelöf Σ-space X (Problem 269) and Gruenhage's theorem on Lindelöf Σ-spaces with a small diagonal (Problem 300).

In Sect. 1.4 we introduce the basic notions of descriptive set theory and give their applications to C_p-theory. This section features three main results: Christensen's theorem on analyticity of $C_p(X)$ (Problem 366), Fremlin's theorem on K-analytic spaces whose compact subspaces are metrizable (Problem 395) and Pytkeev's theorem on condensations of Borel sets (Problem 354).

The first part of Sect. 1.5 comprises some results on decompositions of $C_p(X)$ into a finite or countable union of subspaces with "nice" properties. The second part is devoted to the study of the existence of good mappings between $C_p(X)$ and $C_p(Y)$ and the simplest implications this has for the spaces X and Y. We also have two main results in this section: Gerlits–Pytkeev theorem on k-property in $C_p(X)$ (Problem 465) and Tkachuk's theorem on discreteness of X if $C_p(X)$ is homeomorphic to a retract of a $G_{\delta\sigma}$-subspace of $\mathbb{R}^X$ (Problem 500).

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1.1 Some Additional Axioms and Hereditary Properties

A space X is *left-separated (right-separated)* if there exists a well-order $<$ on X such that the set $\{y \in X : y < x\}$ is closed (open) in X for any $x \in X$. A space X is *scattered* if any subspace $Y \subset X$ has an isolated point. Recall that $\varphi^*(X) = \sup\{\varphi(X^n) : n \in \mathbb{N}\}$ and $h\varphi(X) = \sup\{\varphi(Y) : Y \subset X\}$ for any cardinal invariant φ. All results of this book are proved assuming that ZFC axioms hold. The abbreviation ZFC stands for Zermelo–Fraenkel–Choice. This axiomatic system is the most accepted one at the present moment. We won't need to have the knowledge of what the axioms of ZFC exactly say. It is sufficient to know that all they do is to postulate some very natural properties of sets. For the reader who wants to learn more, the book of Kunen (1980) is an excellent introduction to this subject.

In the twentieth century topologists and set-theorists discovered that there were some very natural problems which could not be solved using ZFC axioms only; to fix this, quite a few additional axioms have been created. Practically all of those axioms are proved to be *consistent with ZFC* which means that if ZFC has no contradiction, then ZFC, together with the axiom in question, does not have one. In this section we formulate the most popular additional axioms and their applications. All results of this book are proved in ZFC if no additional assumptions are formulated explicitly; however, we sometimes emphasize this.

The statement CH (called *Continuum Hypothesis*) says that the first uncountable ordinal is equal to the continuum, i.e., $\omega_1 = \mathfrak{c}$. The statement "$\kappa^+ = 2^\kappa$ for any infinite cardinal κ" is called *Generalized Continuum Hypothesis (GCH)*.

A partial order on a set $\mathcal{P}$ is a relation $\leq$ on $\mathcal{P}$ with the following properties:

(PO1) $p \leq p$ for any $p \in \mathcal{P}$;
(PO2) $p \leq q$ and $q \leq r$ imply $p \leq r$;
(PO3) $p \leq q$ and $q \leq p$ imply $p = q$.

The pair $(\mathcal{P}, \leq)$ is called *a partially ordered set*. If the order is clear, we will write $\mathcal{P}$ instead of $(\mathcal{P}, \leq)$. Let $(\mathcal{P}, \leq)$ be a partially ordered set. The elements $p, q \in P$ are called *compatible* if there is $r \in P$ such that $r \leq p$ and $r \leq q$. If p and q are not compatible, they are called *incompatible*. A set $A \subset \mathcal{P}$ is *an antichain* if the elements of A are pairwise incompatible. We say that $(\mathcal{P}, \leq)$ *has the property ccc* if any antichain of $\mathcal{P}$ is countable. A set $D \subset \mathcal{P}$ is called *dense* in $\mathcal{P}$ if, for every $p \in \mathcal{P}$, there is $q \in D$ such that $q \leq p$.

A non-empty set $F \subset \mathcal{P}$ is *a filter* if it has the following properties:

(F1) for any $p, q \in F$, there is $r \in F$ such that $r \leq p$ and $r \leq q$;
(F2) if $p \in F$ and $p \leq q$, then $q \in F$.

Given an infinite cardinal κ, we denote by $\mathrm{MA}(\kappa)$ the following statement: for any ccc partial order $\mathcal{P}$ and any family $\mathcal{D}$ of dense subsets of $\mathcal{P}$ with $|\mathcal{D}| \leq \kappa$, there is a filter $F \subset \mathcal{P}$ such that $F \cap D \neq \emptyset$ for any $D \in \mathcal{D}$. Now, *Martin's axiom, MA,* says that $\mathrm{MA}(\kappa)$ holds for any infinite $\kappa < \mathfrak{c}$.

A subset $C \subset \omega_1$ is called *club (≡closed and unbounded)* if C is uncountable and closed in the order topology on ω_1. A set $S \subset \omega_1$ is *stationary* if $S \cap C \neq \emptyset$

for any club $C \subset \omega_1$. *Jensen's axiom* $\diamondsuit$ is the statement: for each $\alpha < \omega_1$, there is a set $A_\alpha \subset \alpha$ such that, for any $A \subset \omega_1$, the set $\{\alpha \in \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary. The principle $\diamondsuit^+$ is the following statement: for each $\alpha \in \omega_1$, there is a countable family $\mathcal{A}_\alpha \subset \exp(\alpha)$ such that, for any $A \subset \omega_1$, there is a club $C \subset \omega_1$ for which $A \cap \alpha \in \mathcal{A}_\alpha$ and $C \cap \alpha \in \mathcal{A}_\alpha$ for any $\alpha \in C$. The sequence $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ is called *a* $\diamondsuit^+$*-sequence.*

A space X is called *zero-dimensional* if X has a base consisting of clopen sets. A point $x \in X$ is called *a* P*-point* if any countable intersection of neighborhoods of x is a neighborhood of x. An uncountable dense-in-itself space X is called *Luzin* (also written *Lusin*) if any nowhere dense subspace of X is countable. Say that X is an L*-space* if $hl(X) = \omega < d(X)$; if $hd(X) = \omega < l(X)$, then X is called *an* S*-space*. The axiom SA says that there are no S-spaces, i.e., that every regular hereditarily separable space is Lindelöf. Furthermore, X is *a strong* S*-space* if $hd^*(X) = \omega < l(X)$; if $hl^*(X) = \omega < d(X)$, then X is called *strong* L*-space.*

A tree is a partially ordered set $(\mathcal{T}, \leq)$ such that, for every $x \in \mathcal{T}$, the set $L_x = \{y \in \mathcal{T} : y < x\}$ is well ordered by $\leq$. We will often write $\mathcal{T}$ instead of $(\mathcal{T}, \leq)$. If $\mathcal{T}$ is a tree and $x \in \mathcal{T}$, then *the height* of x in $\mathcal{T}$ or $\mathrm{ht}(x, \mathcal{T})$ is the ordinal isomorphic to L_x. For each ordinal α, the α*-th level* of $\mathcal{T}$ or $\mathrm{Lev}_\alpha(\mathcal{T})$ is the set $\{x \in \mathcal{T} : \mathrm{ht}(x, \mathcal{T}) = \alpha\}$. The height $\mathrm{ht}(\mathcal{T})$ of the tree $\mathcal{T}$ is the least α such that $\mathrm{Lev}_\alpha(\mathcal{T}) = \emptyset$. A subset $\mathcal{T}' \subset \mathcal{T}$ is called *a subtree* of $\mathcal{T}$ if $L_x \subset \mathcal{T}'$ for every $x \in \mathcal{T}'$. A subset $C \in \mathcal{T}$ is called *a chain* if C is linearly ordered by $\leq$, i.e., every two elements of C are comparable. *An antichain* of $\mathcal{T}$ is a set $A \subset \mathcal{T}$ such that $x, y \in A$ and $x \neq y$ implies $x \nleq y$ and $y \nleq x$. For every infinite cardinal κ, a κ*-Souslin tree* is a tree $\mathcal{T}$ such that $|\mathcal{T}| = \kappa$ and every chain and every antichain have cardinality $< \kappa$. An ω_1-Souslin tree is called *Souslin tree.* If κ is a regular cardinal, *a* κ*-tree* is a tree of height κ with levels of cardinality $< \kappa$. *A* κ*-Aronszajn* tree is a κ-tree with no chains of cardinality κ. An ω_1-Aronszajn tree is called *Aronszajn tree.*

If $f : X \to Y$ and $Z \subset X$, we denote the restriction of f to Z by $f|_Z$ or $f|Z$. If we have maps $f, g : X \to Y$, then $f \approx g$ if the set $\{x \in X : f(x) \neq g(x)\}$ is finite. Given functions $f : X \to Y$ and $g : X_1 \to Y_1$, we say that $f \subset g$ if $X \subset X_1$, $Y \subset Y_1$ and $g|X = f$. Now, ω^α is the set of all maps from α to ω and $\omega^{<\omega_1} = \bigcup\{\omega^\alpha : \alpha < \omega_1\}$. Any ω_1-sequence $\{s_\alpha : \alpha < \omega_1\} \subset \omega^{<\omega_1}$ such that $s_\alpha \in \omega^\alpha$ is an injective map and $s_\beta|\alpha \approx s_\alpha$ for all $\alpha < \beta < \omega_1$ is called *Aronszajn coding*. Denote by $\mathbf{P}$ the set of all monotonically increasing functions from ω^ω, i.e., $\mathbf{P} = \{f \in \omega^\omega : f(i) < f(j) \text{ whenever } i < j\}$. Given $f, g \in \omega^\omega$, we say that $f <^* g$ if there exists $m \in \omega$ such that $f(n) < g(n)$ for all $n \geq m$. A sequence $\{f_\alpha : \alpha < \gamma\} \subset \omega^\omega$ is called *strictly* $<^*$*-increasing* if $f_\alpha <^* f_\beta$ for all $\alpha < \beta < \gamma$. A set $S \subset \omega^\omega$ is $<^*$*-cofinal* in ω^ω if, for any $f \in \omega^\omega$, we have $f <^* g$ for some $g \in S$.

001. Given an infinite cardinal κ prove that the following properties are equivalent for any space X:

(i) $hl(X) \leq \kappa$;
(ii) $l(X) \leq \kappa$ and every $U \in \tau(X)$ is a union of $\leq \kappa$-many closed subsets of X;
(iii) $l(X) \leq \kappa$ and every closed $F \subset X$ is a G_κ-set in X;
(iv) $l(U) \leq \kappa$ for any open $U \subset X$.

In particular, a space X is hereditarily Lindelöf if and only if it is Lindelöf and perfect.

002. Prove that a space X is hereditarily normal if and only if any open subspace of X is normal.
003. Prove that if X is perfectly normal, then any $Y \subset X$ is also perfectly normal.
004. Let X be any space. Prove that $hd(X) = \sup\{|A| : A$ is a left-separated subspace of $X\}$. In particular, the space X is hereditarily separable if and only if every left-separated subspace of X is countable.
005. Let X be any space. Prove that $hl(X) = \sup\{|A| : A$ is a right-separated subspace of $X\}$. In particular, the space X is hereditarily Lindelöf if and only if every right-separated subspace of X is countable.
006. Prove that a space is right-separated if and only if it is scattered.
007. Let X be a left-separated space. Prove that $hl(X) \leq s(X)$. In particular, any left-separated space of countable spread is hereditarily Lindelöf.
008. Let X be a right-separated space. Prove that $hd(X) \leq s(X)$. In particular, any right-separated space of countable spread is hereditarily separable.
009. Prove that any space has a dense left-separated subspace.
010. Suppose that $s(X) = \omega$. Prove that X has a dense hereditarily Lindelöf subspace.
011. Prove that for any space X, we have $hl^*(X) = hl(X^\omega)$. In particular, if all finite powers of X are hereditarily Lindelöf, then X^ω is hereditarily Lindelöf.
012. Prove that for any space X, we have $hd^*(X) = hd(X^\omega)$. In particular, if all finite powers of X are hereditarily separable, then X^ω is hereditarily separable.
013. Prove that for any space X, we have $s^*(X) = s(X^\omega)$.
014. Suppose that $s(X \times X) \leq \kappa$. Prove that $hl(X) \leq \kappa$ or $hd(X) \leq \kappa$. In particular, if $s(X \times X) = \omega$, then X is hereditarily separable or hereditarily Lindelöf.
015. Prove that $|X| \leq 2^{hl(X)}$ for any space X. In particular, any hereditarily Lindelöf space has cardinality $\leq \mathfrak{c}$.
016. Prove that $s(X \times X) \leq s(C_p(X)) \leq s^*(X)$ for any space X.
017. Prove that $hd(X \times X) \leq hl(C_p(X)) \leq hd^*(X)$ for any space X.
018. Prove that $hl(X \times X) \leq hd(C_p(X)) \leq hl^*(X)$ for any space X.
019. For an arbitrary $n \in \mathbb{N}$, let $J_n = J(n)$ be the hedgehog with n spines. Prove that $s(X^n) \leq s(C_p(X, J_n)) \leq s(C_p(X) \times C_p(X))$ for any space X.
020. For an arbitrary $n \in \mathbb{N}$, let $J_n = J(n)$ be the hedgehog with n spines. Prove that $hd(X^n) \leq hl(C_p(X, J_n)) \leq hl(C_p(X) \times C_p(X))$ for any space X.

Every problem is short, so it won't be difficult to find a reference in it. An introductory part *is never longer than two pages* so, hopefully, it is not hard to find a reference in it either. Please keep in mind that a solution of a problem can be pretty long, but its definitions *are always given in the beginning*.

The symbols are arranged in alphabetical order; this makes it easy to find the expressions $B(x,r)$ and βX, but it is not immediate what to do if we are looking for $\bigoplus_{t\in T} X_t$. I hope that the placement of the expressions which start with Greek letters or mathematical symbols is intuitive enough to be of help to the reader. Even if it is not, then there are only three pages to plough through. The alphabetic order is *by line* and not by column. For example, the first three lines contain symbols which start with "A" or something similar and lines 3–5 are for the expressions beginning with "B", "β" or "$\mathbb{B}$".

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