Applications of the reflection functors in paratopological groups

M. Tkachenko

Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, C.P. 09340, Del. Iztapalapa, D.F., Mexico

Abstract

We apply $T_i$-reflections for $i = 0, 1, 2, 3$, as well as the regular reflection defined by the author in [20] for the further study of paratopological and semitopological groups. We show that many topological properties are invariant and/or inverse invariant under taking $T_i$-reflections in paratopological groups. Using this technique, we prove that every $\sigma$-compact paratopological group has the Knaster property and, hence, is of countable cellularity. We also prove that an arbitrary product of locally feebly compact paratopological groups is a Moscow space, thus generalizing a similar fact established earlier for products of feebly compact topological groups. The proof of the latter result is based on the fact that the functor $T_2$ of Hausdorff reflection 'commutes' with arbitrary products of semitopological groups. In fact, we show that the functors $T_0$ and $T_1$ also commute with products of semitopological groups, while the functors $T_3$ and Reg commute with products of paratopological groups.

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1. Introduction

Many results concerning paratopological or semitopological groups have been proved assuming certain separation axioms (see [1,3–5,9,13,19]). This is of course not surprising since neither of the implications

$$T_0 \Rightarrow T_1 \Rightarrow T_2 \Rightarrow \text{regular}$$

is valid in the class of paratopological groups, while the validity of the implication

$$\text{regular} \Rightarrow \text{Tychonoff}$$

has very recently been proved by T. Banakh and A. Ravsky in [6].
Ravsky [10] started a revision of the use of separation axioms in the proofs involving paratopological groups. He established, in particular, that every locally compact paratopological group was a topological group, without assuming any separation axiom (see [10, Proposition 5.5]). Romaguera and Sanchis [12] contributed to this revision, while Ravsky exhibited in [11] more cases where a custom use of separation axioms had been excessive.

In [20,21] we present a ‘universal’ approach to analyzing the interaction between many topological properties on the one hand and the usual axioms of separation in the categories of semitopological and paratopological groups on the other hand. It is proved in [20, Proposition 2.5] that for every semitopological group $G$ and every separation axiom $T_i$ with $i \in \{0, 1, 2, 3, r, t\}$, there exists a unique $T_i$-reflection of $G$, i.e. a semitopological group $T_i(G)$ satisfying the $T_i$ separation axiom which maintains a maximal possible amount of information about the original group $G$. To unify our notation, we use $T_r$ and $T_t$ to refer to the regular and Tychonoff separation axiom, respectively. More precisely, the $T_i$-reflection of $G$ is a pair $(T_i(G), \varphi_{G,i})$, where $\varphi_{G,i}$ is a continuous homomorphism of $G$ onto a semitopological group $T_i(G)$ satisfying the $T_i$ separation axiom and having the following important property: Given a continuous mapping $f: G \to X$ to a $T_i$-space $X$, there exists a continuous mapping $g: T_i(G) \to X$ such that $f = g \circ \varphi_{G,i}$.

Informally speaking, the canonical homomorphism $\varphi_{G,i}$ is a right divisor for every continuous mapping of $G$ to a $T_i$-space $X$. Hence every continuous mapping of $G$ to a $T_i$-space $X$ can be reconstructed in a canonical way from a continuous mapping of $T_i(G)$ to $X$. In fact, the $T_i$-reflection gives rise to a covariant functor $T_i$ from the category of semitopological groups to the category of semitopological groups satisfying the $T_i$ separation axiom (see [20, Corollary 2.8]). The morphisms in both categories are continuous homomorphisms. It is also shown in [20] (see also the introduction in [21]) that the functors $T_i$ with $i \in \{0, 1, 2, 3, r\}$ preserve paratopological groups, i.e. $T_i(G)$ is a paratopological group if so is $G$. The same conclusion is valid for the functor $T_t$ since every regular paratopological group is Tychonoff [6]. In fact, the functors $T_r$ and $T_t$ coincide in the category of paratopological groups.

In [20,21] we established a number of properties of the canonical homomorphisms $\varphi_{G,i}$ and the functors $T_i$, for $i \in \{0, 1, 2, 3, r\}$. Here we continue the study in this direction and show in Corollary 3.2 that all the functors $T_i$’s respect open subgroups of semitopological groups. We also prove in Section 3 that the functors $T_0$, $T_1$, and $T_2$ commute with products of semitopological groups, while $T_3$ and $Reg = T_r$ commute with products of paratopological groups.

The reflection functors $T_i$’s are quite useful for the study of semitopological and, especially, paratopological groups. Using Proposition 2.5 of [20], the authors of [23] prove that all concepts of $\mathbb{R}$-factorizability, for $i = 0, 1, 2, 3$, coincide in paratopological groups. In other words, similarly to the case of topological groups, $\mathbb{R}$-factorizability (which refers to a special property of continuous real-valued functions) in paratopological groups does not depend on the separation axioms, even if the axioms appear explicitly in the definition of the concept of $\mathbb{R}$-factorizability.

Here we present more examples of eliminating the separation axioms. According to [3, Corollary 5.7.12], every Hausdorff $\sigma$-compact paratopological group has countable cellularity, which extends the corresponding result established in [17] for topological groups. In fact, almost the same argument shows that the conclusion remains valid for $\sigma$-compact paratopological groups satisfying the $T_1$ separation axiom [19, Theorem 6.12]. Here, in Corollary 2.3, we do the final step by dropping the $T_1$ separation restriction: Every $\sigma$-compact paratopological group has countable cellularity. Our argument is based on the fact that the functor $T_2$ does not change the cellularity of paratopological groups. In turn, the latter fact leans on a special property of the canonical homomorphism $\varphi_{G,2}$, for a paratopological group $G$.

Another example of elimination of separation restrictions is given in Section 3. It is known that an arbitrary product of locally feebly compact Hausdorff paratopological groups is a Moscow space (see [19, Proposition 7.4]). We show in Theorem 3.7 that the Hausdorff separation requirement can be dropped in this result, so any product of locally feebly compact paratopological groups is a Moscow space.
Section 2 contains several results of the following type: For a given topological property \( \mathcal{P} \) and \( i \in \{0, 1, 2, 3, r\} \), a paratopological group \( G \) has \( \mathcal{P} \) iff \( T_i(G) \) has \( \mathcal{P} \). If this happens, we say that the functor \( T_i \) respects property \( \mathcal{P} \). For example, according to Proposition 2.2, the functors \( T_0 \), \( T_1 \), and \( T_2 \) respect the Souslin property. Propositions 2.6 and 2.7 state that the functor \( \text{Reg} = T_r \) (hence each of the functors \( T_i \) for \( i = 0, 1, 2, 3 \)) respects weak Lindelöfness and connectedness. Proposition 2.4 implies that the functors \( T_i \) for \( i = 0, 1, 2, 3 \) and \( \text{Reg} \) respect the Knaster and Shanin properties. The latter result enables us to refine the conclusion of Corollary 2.3 about countable cellularity of \( \sigma \)-compact paratopological groups. We deduce in Theorem 2.5 that every \( \sigma \)-compact paratopological group \( G \) has the Knaster property, i.e. every uncountable family of open sets in \( G \) contains an uncountable subfamily such that every two elements of this subfamily meet each other. It is worth mentioning that in some models of ZFC, even \( \sigma \)-compact topological groups can fail to have the Shanin property [16]. Hence, in a sense, the conclusion of Theorem 2.5 is the best possible.

In Proposition 2.9 and Corollary 2.10 we prove that the functors \( T_i \) for \( i = 0, 1, 2, 3 \) and \( \text{Reg} \) respect bounded and strongly bounded subset of paratopological groups (see Definition 2.8). This fact is applied in the forthcoming article [15] to show that if \( B_i \) is a bounded subset of a paratopological group \( G_i \), where \( i \in I \), then the set \( \prod_{i \in I} B_i \) is bounded in \( \prod_{i \in I} G_i \) provided that each group \( G_i \) is totally \( \omega \)-narrow or precompact. Again, we impose no separation restrictions on the factors \( G_i \)’s.

2. Some applications of the reflection functors

We show in this section that the functors \( T_i \) with \( i \in \{0, 1, 2, 3\} \) and \( \text{Reg} \) respect several topological properties. As a consequence we deduce in Corollary 2.3 and Theorem 2.5 that every \( \sigma \)-compact paratopological group has the Knaster property and, therefore, is countably cellular. We also show that a subset \( B \) of a paratopological group \( G \) is (strongly) bounded in \( G \) iff \( \varphi_{G,r}(B) \) is (strongly) bounded in \( \text{Reg}(G) \) (see Proposition 2.9). Similarly, by Proposition 2.11 and Corollary 2.12, a paratopological group \( G \) is (locally) feebly compact iff so is \( \text{Reg}(G) \).

In what follows we will frequently use several results proved in [21]. For the reader’s convenience we collect some of them in the following theorem. First we recall that a continuous mapping \( f : X \to Y \) is said to be \( d \)-open if \( f(U) \) is a dense subset of an open set in \( Y \), for each open set \( U \) in \( X \).

**Theorem 2.1.** Let \( G \) be a paratopological group and \( \varphi_{G,r} : G \to \text{Reg}(G) \) the canonical surjective homomorphism. Then:

(a) the mapping \( \varphi_{G,r} \) is \( d \)-open (see [21, Prop. 3.1]);
(b) if \( U \) is open in \( G \), then \( \varphi_{G,r}(U) = \varphi_{G,r}^{-1}(U) \) is closed in \( \text{Reg}(G) \) and \( \varphi_{G,r}^{-1}(U) = \varphi_{G,r}(U) \) (see [21, Prop. 3.4 (a)]);
(c) if \( U \) and \( V \) are disjoint open sets in \( G \), then \( \text{Int} \varphi_{G,r}(U) \) and \( \text{Int} \varphi_{G,r}(V) \) are disjoint sets in \( \text{Reg}(G) \) (see [21, Prop. 3.4 (c)]).

We start with the preservation of cellularity. As usual, \( c(H) \) stands for the cellularity of \( H \).

**Proposition 2.2.** The equalities

\[
c(G) = c(T_i(G)) = c(\text{Reg}(G))
\]

are valid for every paratopological group \( G \) and every \( i = 0, 1, 2, 3 \).

**Proof.** It is clear that the groups \( T_i(G) \) with \( i = 0, 1, 2, 3 \) and \( \text{Reg}(G) \) are continuous homomorphic images of \( G \). Hence the cellularity of each of them does not exceed the cellularity of \( G \). Similarly, according to
Propositions 3.3 and 3.5 of [20], $\text{Reg}(G)$ is a continuous homomorphic image of $T_i(G)$ for each $i = 0, 1, 2, 3$. Hence the cellularity of $\text{Reg}(G)$ is not greater than the cellularity of each of the groups $G$ and $T_i(G)$, for $i = 0, 1, 2, 3$. It remains to show that $c(G) \leq c(\text{Reg}(G))$.

Suppose that $\gamma$ is a pairwise disjoint family of non-empty open sets in $G$. Let $\varphi_{G,r}: G \to \text{Reg}(G)$ be the canonical surjective homomorphism. For every $U \in \gamma$, let $V_U = \text{Int} \varphi_{G,r}(U)$, where the closure and interior are taken in $\text{Reg}(G)$. It follows from (a) of Theorem 2.1 that the mapping $\varphi_{G,r}$ is $d$-open, so the sets $V_U$ are non-empty. By (c) of Theorem 2.1, the family $\lambda = \{V_U : U \in \gamma\}$ of open sets in $\text{Reg}(G)$ is pairwise disjoint. Since $|\lambda| = |\gamma|$, we conclude that $c(G) \leq c(\text{Reg}(G))$. \qed

It was proved by the author in [17] that every $\sigma$-compact topological group had countable cellularity. Reznichenko extended this result to $\sigma$-compact Hausdorff paratopological groups (see [4, Corollary 5.7.12]). Afterwards it was shown in [19, Theorem 6.12] that the Hausdorff separation property could be weakened to the $T_1$ separation axiom. Proposition 2.2 enables us to eliminate these separation restrictions completely:

**Corollary 2.3.** Every $\sigma$-compact paratopological group has countable cellularity.

**Proof.** Let $\varphi_{G,2}$ be the canonical homomorphism of a $\sigma$-compact paratopological group $G$ onto $T_2(G)$. Since $\varphi_{G,2}$ is continuous, the Hausdorff paratopological group $T_2(G)$ is $\sigma$-compact. Hence the cellularity of $T_2(G)$ is countable, by [4, Corollary 5.7.12], and the same conclusion is valid for $G$ according to Proposition 2.2. \qed

Let us recall that a space $X$ has the Knaster property if every uncountable family $\gamma$ of open sets in $X$ contains an uncountable subfamily $\lambda$ such that $U \cap V \neq \emptyset$ for all $U, V \in \lambda$. It is clear that a space with the Knaster property has countable cellularity. Similarly, a space $X$ has the Shanin property if every uncountable family $\gamma$ of open sets in $X$ contains an uncountable subfamily with the finite intersection property. Evidently, the Shanin property implies the Knaster property. It is also known that both properties are productive [22] and stable with respect to taking continuous images.

It turns out that the functors $T_i$ for $i = 0, 1, 2, 3$ and $\text{Reg}$ respect the Knaster and Shanin properties in the following sense:

**Proposition 2.4.** The following conditions are equivalent for an arbitrary paratopological group $G$:

(a) $G$ has the Knaster (Shanin) property;
(b) $T_0(G)$ has the Knaster (Shanin) property;
(c) $T_1(G)$ has the Knaster (Shanin) property;
(d) $T_2(G)$ has the Knaster (Shanin) property;
(e) $T_3(G)$ has the Knaster (Shanin) property;
(f) $\text{Reg}(G)$ has the Knaster (Shanin) property.

**Proof.** We prove the proposition for the Shanin property, since the argument for the Knaster property is similar and becomes even simpler.

Since the Shanin property is invariant under continuous onto mappings, the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (f)$ follow from [20, Proposition 3.5]. Similarly, we have that $(a) \Rightarrow (e) \Rightarrow (f)$, by [20, Proposition 3.7]. It remains to show that $(f)$ implies $(a)$.

Suppose that the space $\text{Reg}(G)$ has the Shanin property. Let $\varphi_{G,r}$ be the canonical homomorphism of $G$ onto $\text{Reg}(G)$. Consider a family $\{U_\alpha : \alpha \in \omega_1\}$ of non-empty open sets in $G$. For every $\alpha \in \omega_1$, put $V_\alpha = \text{Int} \varphi_{G,r}(U_\alpha)$, where the closure and interior are taken in $\text{Reg}(G)$. According to (a) of Theorem 2.1, each $V_\alpha$ is a non-empty open subset of $\text{Reg}(G)$. 

By our assumption about $Reg(G)$, there exists an uncountable set $A \subseteq \omega_1$ such that the family $\{V_\alpha : \alpha \in A\}$ has the finite intersection property. Then [21, Proposition 3.4(b)] implies that the family $\{U_\alpha : \alpha \in A\}$ has the finite intersection property as well. □

Making use of Proposition 2.4, we can strengthen the conclusion of Corollary 2.3.

Theorem 2.5. Every $\sigma$-compact paratopological group has the Knaster property.

Proof. Let $G$ be an arbitrary $\sigma$-compact paratopological group. The canonical homomorphism $\varphi_{G,2}: G \to T_2(G)$ is continuous and surjective, so the Hausdorff paratopological group $T_2(G)$ is also $\sigma$-compact. By Proposition 2.4, it suffices to verify that the group $H = T_2(G)$ has the Knaster property.

According to [4, Lemma 5.7.10], there exists a topological group $T$ homeomorphic to a closed subspace of $H \times H$ such that $H$ is a continuous homomorphic image of $T$. Evidently, the group $T$ is $\sigma$-compact. Hence $T$ has the Knaster property by [4, Corollary 5.4.8]. Thus $H$ also has the Knaster property as a continuous image of $T$. □

It is natural to ask whether one can strengthen the conclusion of Theorem 2.5 by replacing the Knaster property with the Shanin property. Shakhmatov proved in [16] that this is impossible since ZFC is consistent with the existence of a $\sigma$-compact Hausdorff topological group which fails to have the Shanin property.

A space $X$ is weakly Lindelöf if every open covering of $X$ contains a countable subfamily whose union is dense in $X$. Once again, the proof of the following fact leans on Theorem 2.1.

Proposition 2.6. A paratopological group $G$ is weakly Lindelöf iff so is $Reg(G)$.

Proof. Let $\varphi_{G,r}$ be the canonical homomorphism of $G$ onto $Reg(G)$. If $G$ is weakly Lindelöf, then the continuity of $\varphi_{G,r}$ implies that $Reg(G)$ is also weakly Lindelöf.

Conversely, suppose that the group $Reg(G)$ is weakly Lindelöf and consider an open covering $\gamma$ of $G$. The mapping $\varphi_{G,r}$ is $d$-open by (a) of Theorem 2.1, so the set $O_U = \text{Int}f(U)$ is open in $Reg(G)$ and contains $f(U)$, for every open set $U$ in $G$. Let $\lambda = \{O_U : U \in \gamma\}$. Then $\lambda$ is an open covering of $Reg(G)$, so it contains a countable subfamily $\lambda'$ such that $\bigcup \lambda'$ is dense in $Reg(G)$. Take a countable subfamily $\gamma'$ of $\gamma$ such that $\gamma' = \{O_U : U \in \gamma'\}$. Then (c) of Theorem 2.1 implies that $\bigcup \gamma'$ is dense in $G$. Hence $G$ is weakly Lindelöf. □

Connectedness behaves similarly to weak Lindelöfness with respect to the functor $Reg$.

Proposition 2.7. A paratopological group $G$ is connected iff $Reg(G)$ is connected.

Proof. Since $Reg(G)$ is a continuous homomorphic image of $G$, it suffices to verify that $Reg(G)$ is disconnected provided that $G$ is disconnected. Suppose that $G$ is the union of disjoint non-empty open sets $U$ and $V$. Since $U$ and $V$ are closed in $G$, (a)–(c) of Theorem 2.1 imply that $\varphi_{G,r}(U)$ and $\varphi_{G,r}(V)$ are disjoint closed subsets of $Reg(G)$ which cover $Reg(G)$. So the group $Reg(G)$ is disconnected. □

In the rest of this section we consider bounded and strongly bounded subsets of paratopological groups, as well as feebly compact paratopological groups.

Definition 2.8. Let $B$ be a non-empty subset of a space $X$.

(a) The set $B$ is called bounded in $X$ if every locally finite family of open sets in $X$ contains at most finitely many elements which intersect $B$. 

(b) The set $B$ is said to be strongly bounded in $X$ if every infinite family of open sets in $X$ each of which meets $B$, contains an infinite subfamily $\{U_n : n \in \omega\}$ satisfying the following property:

(*) For every filter $\mathcal{F}$ of infinite subsets of $\omega$, the set $\bigcap_{F \in \mathcal{F}} \bigcup_{n \in F} U_n$ is non-empty.

It is clear that every strongly bounded subset of a space $X$ is bounded in $X$, but the converse is false [8]. We also recall that a space $X$ is feebly compact if it is bounded in itself, i.e. every infinite family of open sets in $X$ has an accumulation point.

Similarly to the Knaster and Shanin properties, bounded and strongly bounded subsets of paratopological groups are well behaved with respect to the functors $\text{Reg}$ and $T_i$ with $i = 0, 1, 2, 3$.

**Proposition 2.9.** Let $\varphi_{G, r}$ be the canonical homomorphism of a paratopological group $G$ onto $\text{Reg}(G)$. Then $\varphi_{G, r}^{-1}(C)$ is (strongly) bounded in $G$, for every (strongly) bounded subset $C$ of $\text{Reg}(G)$. In particular, $\varphi_{G, r}^{-1}\varphi_{G, r}(B)$ is (strongly) bounded in $G$ for every (strongly) bounded subset $B$ of $G$.

**Proof.** We start with the case of bounded subsets. Let $C$ be a non-empty subset of $\text{Reg}(G)$. Suppose that the set $B = \varphi_{G, r}^{-1}(C)$ is not bounded in $G$. Then there exists an infinite locally finite family $\{U_n : n \in \omega\}$ of open sets in $G$ such that each $U_n$ meets $B$. For every $n \in \omega$, let $V_n = \text{Int} \varphi_{G, r}(U_n)$. Since the mapping $\varphi_{G, r}$ is $d$-open according to (a) of Theorem 2.1, we see that $\varphi_{G, r}(U_n)$ is a subset of $V_n$ for each $n \in \omega$. Hence each $V_n$ meets $C$.

We claim that the family $\{V_n : n \in \omega\}$ is locally finite in $\text{Reg}(G)$. Indeed, take an arbitrary point $y \in \text{Reg}(G)$ and pick $x \in G$ with $\varphi_{G, r}(x) = y$. There exists an open neighborhood $U$ of $x$ in $G$ which intersects only finitely many of the sets $U_n$’s. Then (c) of Theorem 2.1 implies that the open neighborhood $V = \text{Int} \varphi_{G, r}(U)$ of $y$ in $\text{Reg}(G)$ meets at most finitely many elements of the family $\{V_n : n \in \omega\}$, whence our claim follows. Therefore, the set $C$ is not bounded in $\text{Reg}(G)$. This proves the first part of the proposition for bounded sets.

Suppose that $C$ is a strongly bounded subset of $\text{Reg}(G)$ and let $B = \varphi_{G, r}^{-1}(C)$. Consider an infinite family $\{U_n : n \in \omega\}$ of open sets in $G$ each of which meets $B$ and for every $n \in \omega$, let $V_n = \text{Int} \varphi_{G, r}(U_n)$. As in the first part of the proof, every $V_n$ intersects $C$. Let $\mathcal{F}$ be a filter of infinite subsets of $\omega$. Since $C$ is strongly bounded in $\text{Reg}(G)$, there exists a point $y_0 \in \text{Reg}(G)$ such that every neighborhood of $y_0$ meets infinitely many elements of the family $\{V_n : n \in F\}$, for each $F \in \mathcal{F}$. Take a point $x_0 \in G$ with $\varphi_{G, r}(x_0) = y_0$. We claim that the same happens for every open neighborhood $U$ of $x_0$ in $G$ with respect to the family $\{U_n : n \in \omega\}$.

Indeed, let $V = \text{Int} \varphi_{G, r}(U)$. Then $V$ is an open neighborhood of $y_0$ in $\text{Reg}(G)$, so $V \cap V_n \neq 0$ for infinitely many $n \in F$, where $F$ is an arbitrary element of $\mathcal{F}$. It now follows from (c) of Theorem 2.1 that $U \cap U_n \neq 0$ iff $V \cap V_n \neq 0$, for each $n \in \omega$. Hence the family $\{U_n : n \in F\}$ accumulates at $x_0$, for each $F \in \mathcal{F}$, and $B$ is strongly bounded in $G$.

Finally, if $B$ is (strongly) bounded in $G$, then $\varphi_{G, r}(B)$ is (strongly) bounded in $\text{Reg}(G)$ and, therefore, $\varphi_{G, r}^{-1}\varphi_{G, r}(B)$ is (strongly) bounded in $G$. \hfill $\square$

According to Propositions 3.5 and 3.7 in [20], for every $i = 0, 1, 2, 3$, there exists a continuous surjective homomorphism $\psi_i : T_i(G) \to \text{Reg}(G)$ satisfying $\varphi_{G, r} = \psi_i \circ \varphi_{G, i}$. Combining this fact with Proposition 2.9 we obtain:

**Corollary 2.10.** Let $\varphi_{G, i} : G \to T_i(G)$ be the canonical homomorphism of a paratopological group $G$ onto $T_i(G)$, where $i \in \{0, 1, 2, 3\}$. Then a set $B \subseteq G$ is (strongly) bounded in $G$ if and only if $\varphi_{G, i}(B)$ is (strongly) bounded in $T_i(G)$.

The next fact follows from Proposition 2.9 and Corollary 2.10.
Proposition 2.11. The following conditions are equivalent for an arbitrary paratopological group $G$:

(a) $G$ is feebly compact;
(b) $T_0(G)$ is feebly compact;
(c) $T_1(G)$ is feebly compact;
(d) $T_2(G)$ is feebly compact;
(e) $T_3(G)$ is feebly compact;
(f) $\text{Reg}(G)$ is feebly compact.

A space $X$ is called locally feebly compact if every point $x \in X$ has a feebly compact neighborhood. Equivalently, $X$ is locally feebly compact if for every $x \in X$, there exists an open neighborhood $U$ of $x$ in $X$ such that $\overline{U}$ is feebly compact.

Slightly modifying the argument in the proof of Proposition 2.9, one deduces the following fact whose proof is hence omitted:

Corollary 2.12. The following conditions are equivalent for a paratopological group $G$:

(a) $G$ is locally feebly compact;
(b) $T_2(G)$ is locally feebly compact;
(c) $\text{Reg}(G)$ is locally feebly compact.

Corollary 2.12 will be used in the proof of Theorem 3.7 in Section 3.

Let us recall that a subset $B$ of a space $X$ is relatively pseudocompact in $X$ if every infinite family $\gamma$ of open sets in $X$ meeting $B$ has an accumulation point in $B$ (see [2,8]). It is clear that every relatively pseudocompact subset of $X$ is bounded in $X$, but not vice versa. It is also known that every relatively pseudocompact subset of a topological group $G$ is strongly bounded in $G$. Indeed, a relatively pseudocompact subset $B$ of $G$ is bounded in $G$, while Lemmas 2.8 and 2.10 of [18] together imply that every bounded subset of $G$ is strongly bounded.

The proof of the following fact is omitted since it is similar to the argument in the proof of Proposition 2.9.

Proposition 2.13. A subset $B$ of a paratopological group $G$ is relatively pseudocompact in $G$ iff $\varphi_{G,r}(B)$ is relatively pseudocompact in $\text{Reg}(G)$.

We recall that a paratopological group $G$ is called 2-pseudocompact if the intersection $\bigcap_{n \in \omega} \overline{U_n^{-1}}$ is not empty, for every decreasing sequence $\{U_n : n \in \omega\}$ of non-empty open sets in $G$.

Example 2.14 below shows that Proposition 2.11 is no more valid either for the Baire property or 2-pseudocompactness in place of feebly compactness, even if we restrict ourselves to considering the $T_1$-reflection only.

Example 2.14. There exists a $\sigma$-compact, feebly compact, second countable paratopological group $G$ satisfying the $T_0$ separation axiom such that $T_1(G)$ is the trivial one-element group, but $G$ is neither 2-pseudocompact nor Baire.

Proof. Let $\mathbb{R}$ be the additive group of the reals. Denote by $\tau$ the topology on $\mathbb{R}$ whose base consists of the sets $(x, \infty)$, with $x \in \mathbb{R}$. Then $G = (\mathbb{R}, \tau)$ is a $T_0$ paratopological group with a countable base. It is clear that each of the sets $[x, \infty)$ is a compact subspace of $G$, so $G$ is $\sigma$-compact. It is also clear that every non-empty open set is dense in $G$, so $G$ is feebly compact. Since the closure of the singleton $\{0\}$ in $G$ is the set $(-\infty, 0]$, the minimal closed subgroup of $G$ containing 0 is the whole group $G$. Hence the group $T_1(G)$ is trivial, by [20, Theorem 3.1].
Let us verify that $G$ is neither $2$-pseudocompact nor Baire. For every positive integer $n$, let $U_n = (n, \infty)$. Then $\{U_n : n \in \mathbb{N}^+\}$ is a decreasing sequence of non-empty open sets in $G$. An easy verification shows that $\bigcap_{n=1}^{\infty} U_n = (-\infty, -n]$, so the intersection $\bigcap_{n=1}^{\infty} U_n$ is empty. Hence $G$ fails to be $2$-pseudocompact. It is also clear that each set $(-\infty, n]$ is closed and nowhere dense in $G$, so $G$ is not Baire. \[\square\]

The \textit{index of narrowness} of a semitopological (paratopological) group $G$, denoted by $in(G)$, is the minimum cardinal number $\tau$ such that for every neighborhood $U$ of the neutral element in $G$, there exists a subset $C$ of $G$ such that $UC = G = CU$ and $|C| \leq \tau$. The following example shows that the functor $T_1$ does not respect the Lindelöf property or the index of narrowness.

\textbf{Example 2.15.} For every infinite regular cardinal $\tau$, there exists a $T_0$ paratopological group $G$ such that $l(G) = \tau$ and $in(G) = \tau$, while the group $T_1(G)$ is trivial, i.e. $T_1(G)$ is a singleton.

\textbf{Proof.} Let $\tau$ be an infinite regular cardinal. According to [7], there exists a linearly ordered field $(F, +, \cdot, <)$ such that the cofinality of $(F, <)$ is equal to $\tau$. Then the family $\{[x, \infty) : x \in F\}$ is a base for a topology $\mathcal{T}$ on $F$ satisfying the $T_0$ separation axiom, and $G = (F, +, \mathcal{T})$ is an Abelian paratopological group. Since $c(f(F, >) = c(f(F, <) = \tau$, we see that $l(G) = \tau$ and $in(G) = \tau$.

As in Example 2.14, it is easy to verify that the group $T_1(G)$ is trivial. \[\square\]

3. Subgroups, products, and the functors $T_i$

Given a subgroup $H$ of a semitopological (paratopological) group $G$, it is natural to ask whether $T_i(H)$ is topologically isomorphic to a subgroup of $T_i(G)$. More precisely, we are interested in finding out whether the restriction of the canonical homomorphism $\varphi_{G,i}$ to $H$ is a topological isomorphism of $H$ onto the subgroup $\varphi_{G,i}(H)$ of $T_i(G)$, where $i \in \{0, 1, 2, 3\}$. If this happens for all $G$ and $H$, we say that the functor $T_i$ respects subgroups of semitopological (paratopological) groups. The same terminology applies to the functors $Reg = T_r$ and $Tych = T_t$.

It is not difficult to present counterexamples to this general problem for each $i = 1, 2, 3$ and for the functors $Reg$, $Tych$ (see Examples 3.13 and 3.14 in [21]). In some special cases, however, the problem is solved affirmatively. For example, it is shown in [21, Proposition 3.11] that the functor $T_0$ respects arbitrary subgroups of semitopological groups, while Lemma 3.7 of [21] states that the functor $T_1$ respects \textit{closed} subgroups of semitopological groups. Further, Theorem 3.12 of [21] establishes that the functors $T_3$ and $Reg$ respect dense subgroups of paratopological groups.

Our aim is to show in Corollary 3.2 below that all the functors $T_i$ for $i = 0, 1, 2, 3$, as well as $Reg$ and $Tych$ respect open subgroups of semitopological groups (for $i = 0, 1$, this follows from the aforementioned results in [21]). In fact, we prove a more general result in Proposition 3.1 for subgroups which are retracts of enveloping groups.

Let us recall that a subset $Y$ of a space $X$ is a \textit{retract} of $X$ if there exists a continuous mapping $r : X \to X$ such that $r(X) = Y$ and $r(x) = x$, for each $x \in Y$.

In Proposition 3.1 below we use the terminology from [20]. In particular, a \textit{PS-class} of spaces is a class which contains a one-point space and is closed under taking arbitrary products and arbitrary subspaces. According to Propositions 2.6 and 2.7 of [20], every $PS$-class $\mathcal{C}$ gives rise to a covariant functor $\mathcal{C}$ in the category of semitopological groups. The functors $T_i$ with $i \in \{0, 1, 2, 3\}$ as well as $Reg$ and $Tych$ correspond to easily identified $PS$-classes.

\textbf{Proposition 3.1.} Let $H$ be a subgroup of a semitopological group $G$ and $\mathcal{C}$ be a $PS$-class of spaces. If $H$ is a retract of $G$, then $\mathcal{C}(H)$ is topologically isomorphic to the subgroup $\varphi_{\mathcal{C}, i}(H)$ of $\mathcal{C}(G)$, where $\varphi_{\mathcal{C}, i} : G \to \mathcal{C}(G)$ is the canonical homomorphism.
Proof. Let \( i_H \) be the identity embedding of \( H \) to \( G \) and \( g = \mathcal{C}(i_H) \). In other words, \( g: \mathcal{C}(H) \to \mathcal{C}(G) \) is a continuous homomorphism satisfying \( \varphi_G^c \mid H = \varphi_G^c \circ i_H = g \circ \varphi_H^c \). We claim that \( g \) is a topological isomorphism of \( \mathcal{C}(H) \) onto the subgroup \( \varphi_G^c(H) \) of \( \mathcal{C}(G) \).

Indeed, let \( f: H \to X \) be an arbitrary continuous mapping to a space \( X \in \mathcal{C} \). If \( r \) is a retraction of \( G \) onto \( H \), then \( \tilde{f} = f \circ r \) is a continuous mapping of \( G \) to \( X \) and \( \tilde{f} \mid H = f \). By the definition of \( \mathcal{C}(G) \), there exists a continuous mapping \( \tilde{h}: \mathcal{C}(G) \to X \) satisfying \( \tilde{f} = \tilde{h} \circ \varphi_G^c \).

\[
\begin{array}{ccc}
H & \xrightarrow{i_H} & G \\
\downarrow {\varphi_H^c} & & \downarrow {\varphi_G^c} \\
\mathcal{C}(H) & \xrightarrow{g} & \mathcal{C}(G) \\
\end{array}
\]

Let \( h = \tilde{h} \mid \varphi_G^c(H) \). It follows from \( \tilde{f} = \tilde{h} \circ \varphi_G^c \) that
\[
f = \tilde{f} \mid H = \tilde{h} \circ \varphi_G^c \mid H = h \circ (\varphi_G^c \mid H).
\]

Since the subgroup \( \varphi_G^c(H) \) of \( \mathcal{C}(G) \) is in \( \mathcal{C} \), we conclude that the pair \((\varphi_G^c(H), \varphi_G^c \mid H)\) is the \( \mathcal{C} \)-reflection of \( H \). We also know, by [20, Proposition 2.2], that all \( \mathcal{C} \)-reflections of the group \( H \) are equivalent. More exactly, there exists a topological isomorphism \( q: \mathcal{C}(H) \to \varphi_G^c(H) \) such that \( \varphi_G^c \mid H = q \circ \varphi_H^c \). Combining the latter equality with \( \varphi_G^c \mid H = g \circ \varphi_H^c \), we infer that \( g = q \). Hence \( g \) is a topological isomorphism of \( \mathcal{C}(H) \)
onto \( \varphi_G^c(H) \). \( \Box \)

Corollary 3.2. If \( H \) is an open subgroup of a semitopological group \( G \), then \( T_i(H) \) is topologically isomorphic to the open subgroup \( \varphi_{G,i}(H) \) of the group \( T_i(G) \), for each \( i \in \{0, 1, 2, 3, r, t\} \).

Proof. Since the open subgroup \( H \) of \( G \) is a retract of \( G \), Proposition 3.1 implies that the subgroup \( \varphi_{G,i}(H) \) of the group \( T_i(G) \) is topologically isomorphic to the group \( T_i(H) \), for each \( i \in \{0, 1, 2, 3, r, t\} \). It remains to verify that \( \varphi_{G,i}(H) \) is open in \( T_i(G) \).

Consider the following commutative diagram, where \( i_H \) is the identity embedding of \( H \) to \( G \) and \( g = T_i(i_H) \).

\[
\begin{array}{ccc}
H & \xrightarrow{i_H} & G \\
\downarrow {\varphi_{H,i}} & & \downarrow {\varphi_{G,i}} \\
T_i(H) & \xrightarrow{g} & T_i(G) \\
\end{array}
\]

Denote by \( X_0 \) the connected two-point space \( \{0, 1\} \), where the singleton \( \{1\} \) is open and dense in \( X_0 \). Let also \( X_i \) be the discrete space \( \{0, 1\} \) for each \( i \in \{1, 2, 3, r, t\} \). It is clear that \( X_i \) satisfies the \( T_i \) separation axiom, where \( i \in \{0, 1, 2, 3, r, t\} \). Consider a mapping \( f \) of \( G \) to \( X_i \) such that \( f(x) = 1 \) iff \( x \in H \). Since every open subgroup of \( G \) is closed, the mapping \( f \) is continuous. According to the definition of \( T_i(G) \), there exists a continuous mapping \( h: T_i(G) \to X_i \) satisfying \( f = h \circ \varphi_{G,i} \). It is clear that for every \( y \in T_i(G) \), \( h(y) = 1 \) iff \( y \in \varphi_{G,i}(H) \). Hence the continuity of \( h \) implies that the subgroup \( \varphi_{G,i}(H) \) of \( T_i(G) \) is open. \( \Box \)

In the following proposition we establish that the functor of \( T_0 \)-reflection ‘commutes’ with the product operation in the category of semitopological groups.
Proposition 3.3. Let $\Pi = \prod_{i \in I} G_i$ be the product of a family of semitopological groups. Then $T_0(\Pi) \cong \prod_{i \in I} T_0(G_i)$.

Proof. Let $\varphi_{i,0}$ be the canonical homomorphism of $G_i$ onto $T_0(G_i)$, for each $i \in I$. Let also $\varphi_0: \Pi \to \prod_{i \in I} T_0(G_i)$ be the Cartesian product of the homomorphisms $\varphi_{i,0}$ with $i \in I$. Since each $\varphi_{i,0}$ is open, continuous, and surjective by [20, Proposition 2.5], so is $\varphi_0$. The product $P = \prod_{i \in I} T_0(G_i)$ is a $T_0$-space, so [20, Theorem 3.1] implies that there exists a continuous homomorphism $p: T_0(\Pi) \to P$ such that $\varphi_0 = p \circ \varphi_{\Pi,0}$, where $\varphi_{\Pi,0}: \Pi \to T_0(\Pi)$ is the canonical homomorphism. In particular, this means that the kernel of $\varphi_{\Pi,0}$ is contained in the kernel of $\varphi_0$. Let us verify the inverse inclusion.

For every $i \in I$, denote by $N_i$ the kernel of $\varphi_{i,0}$. It was shown in [20, Theorem 3.1] that $N_i = K_i \cap (K_i)^{-1}$, where $K_i$ is the intersection of the family of all open neighborhoods of the identity $e_i$ in $G_i$. Similarly, the kernel $C$ of $\varphi_{\Pi,0}$ is $K \cap K^{-1}$, where $K$ is the intersection of all open neighborhoods of the identity $e$ in $\Pi$. It follows from the definition of the homomorphism $\varphi_0$ that its kernel is $N = \prod_{i \in I} N_i$. Let $U$ be an open neighborhood of $e$ in $\Pi$. To finish the proof, it suffices to show that $N \subseteq U \cap U^{-1}$. Since $N$ is symmetric, the latter inclusion is equivalent to the simpler one, $N \subseteq U$.

We can assume without loss of generality that $U$ is a canonical open set in $\Pi$. Hence $U = \prod_{i \in I} U_i$, where each $U_i$ is an open neighborhood of $e_i$ in $G_i$. Our definition of the sets $N_i$’s implies that $N_i \subseteq U_i$ for each $i \in I$, whence the required inclusion $N = \prod_{i \in I} N_i \subseteq U$ follows. This proves that the kernels of the homomorphisms $\varphi_0$ and $\varphi_{\Pi,0}$ coincide. Since both $\varphi_0$ and $\varphi_{\Pi,0}$ are continuous, open, and surjective, we conclude that $T_0(\Pi) \cong \prod_{i \in I} T_0(G_i)$. \qed

The proof of the fact that the functor of $T_1$-reflection ‘commutes’ with products of semitopological groups is slightly different since the kernel of the canonical homomorphism $\varphi_{G,1}: G \to T_1(G)$ differs substantially from the kernel of the homomorphism $\varphi_{G,0}$.

Proposition 3.4. Let $\Pi = \prod_{i \in I} G_i$ be a product of semitopological groups. Then $T_1(\Pi) \cong \prod_{i \in I} T_1(G_i)$.

Proof. For every $i \in I$, denote by $\varphi_{i,1}$ the canonical quotient homomorphism of $G_i$ onto $T_1(G_i)$ and let $\varphi_i: \Pi \to \prod_{i \in I} T_1(G_i)$ be the Cartesian product of the homomorphisms $\varphi_{i,1}$ with $i \in I$. Since each $\varphi_{i,1}$ is open, continuous, and surjective, so is $\varphi_i$. Clearly $P = \prod_{i \in I} T_1(G_i)$ is a $T_1$-space, so $N = \ker \varphi_i$ is a closed invariant subgroup of $\Pi$. It follows from the definition of $\varphi_i$ that $N = \prod_{i \in I} N_i$, where $N_i$ is the kernel of $\varphi_{i,1}$, for each $i \in I$.

Denote by $\varphi_{\Pi,1}$ the canonical homomorphism of $\Pi$ onto $T_1(\Pi)$. Since $\varphi_{\Pi,1}$ is continuous, open and surjective, it suffices to verify that $\ker \varphi_{\Pi,1} = N$. By [20, Theorem 3.1], there exists a continuous homomorphism $p: T_1(G) \to P$ such that $\varphi_i = p \circ \varphi_{\Pi,1}$. This implies that $\ker \varphi_{\Pi,1} \subseteq N$. Let us verify the inverse inclusion.

In what follows we identify each factor $G_i$ with the corresponding subgroup of $\Pi$. As $T_1(\Pi)$ is a $T_1$-space, the group $N_i = \ker \varphi_{i,1}$ is contained in the closed subgroup $\ker \varphi_{\Pi,1}$ of $\Pi$. Since this inclusion is valid for each $i \in I$, we see that the direct sum $\oplus_{i \in I} N_i$ is contained in $\ker \varphi_{\Pi,1}$. However, the group $\oplus_{i \in I} N_i$ is dense in $N = \prod_{i \in I} N_i$, while the group $\ker \varphi_{\Pi,1}$ is closed in $\Pi$ and in $N$. Hence $\ker \varphi_{\Pi,1} = N$. This completes the proof. \qed

Slightly modifying the argument in the proof of Proposition 3.4 we obtain the following fact whose proof is omitted:

Proposition 3.5. Let $\Pi = \prod_{i \in I} G_i$ be a product of semitopological groups. Then $T_2(\Pi) \cong \prod_{i \in I} T_2(G_i)$.

Seemingly, the functors $T_3$ and $Reg$ interact with products in a similar way, but we can prove this only in the narrower category of paratopological groups.
Proposition 3.6. Let $\Pi = \prod_{i \in I} G_i$ be a product of paratopological groups. Then $T_3(\Pi) \cong \prod_{i \in I} T_3(G_i)$ and $\text{Reg}(\Pi) \cong \prod_{i \in I} \text{Reg}(G_i)$.

Proof. First we consider the functor $T_3$. By [21, Theorem 2.6], $T_3(G)$ is the semiregularization of $G$, for an arbitrary paratopological group $G$. In other words, $\varphi_{G,3}$ is the identity mapping of $G$ onto itself and the family of regular open sets in $G$ forms a base for the topology of $T_3(G)$. According to [11, Lemma 17], the semiregularization of the product of a family of paratopological groups is topologically isomorphic to the product of semiregularizations of the groups in this family. This proves that $T_3(\Pi) \cong \prod_{i \in I} T_3(G_i)$.

The argument in the case of the functor $\text{Reg}$ is similar. It was established in [21, Corollary 2.9] that for every paratopological group $G$, $\text{Reg}(G)$ is topologically isomorphic to the semiregularization of the group $T_3(G)$. By Proposition 3.5, the functor $T_2$ commutes with products of paratopological groups. Applying [11, Lemma 17] once again, we obtain the required conclusion. □

It is known that any product of feebly compact paratopological groups is feebly compact [11, Proposition 22]. The analogue of this fact is clearly false for infinite products of locally feebly compact paratopological groups. However, products of locally feebly compact paratopological groups are always Moscow spaces (a space $X$ is Moscow if the closure of every open set in $X$ is the union of a family of $G_\delta$-sets):

Theorem 3.7. A finite product of locally feebly compact paratopological groups is locally feebly compact. An arbitrary product of locally feebly compact paratopological groups is a Moscow space.

Proof. Let $G = \prod_{i=1}^n G_i$ be a product of locally feebly compact paratopological groups. Proposition 3.5 implies that $T_2(G) = \prod_{i=1}^n T_2(G_i)$. By Corollary 2.12, each factor $T_2(G_i)$ is locally feebly compact. Since the factors $T_2(G_i)$ are Hausdorff, the product group $T_2(G)$ is also locally feebly compact by [19, Proposition 7.4]. Applying Corollary 2.12 once again, we conclude that $G$ is locally feebly compact as well.

To prove the second part of the theorem, we argue in a similar way. Suppose that $G = \prod_{i \in I} G_i$ is the product of a family of locally feebly compact paratopological groups. Then $T_2(G) = \prod_{i \in I} T_2(G_i)$, by Proposition 3.5. Each factor $T_2(G_i)$ is a locally feebly compact Hausdorff paratopological group, so [19, Proposition 7.4] implies that the product group $T_2(G)$ is a Moscow space.

Let $\varphi_{G,2} : G \to T_2(G)$ be the canonical quotient homomorphism. Take an arbitrary non-empty open set $U \subseteq G$. Then $\overline{U} = \varphi_{G,2}^{-1}(\varphi_{G,2}(U))$, according to [21, Proposition 1.2]. Since the homomorphism $\varphi_{G,2}$ is continuous and open, the set $\varphi_{G,2}(U)$ is closed in $T_2(G)$ and $\varphi_{G,2}(U)$ is open in $T_2(G)$. Hence $\varphi_{G,2}(U) = \varphi_{G,2}(\overline{U})$. We know that $T_2(G)$ is a Moscow space, so $\varphi_{G,2}(\overline{U})$ is the union of a family of $G_\delta$-sets in $T_2(G)$. Hence the set $\overline{U} = \varphi_{G,2}^{-1}(\varphi_{G,2}(U))$ is the union of a family of $G_\delta$-sets in $G$. This proves that $G$ is a Moscow space. □

4. Open problems

We know, by Propositions 3.3, 3.4, and 3.5 that the functors $T_0$, $T_1$, and $T_2$ commute with arbitrary products of semitopological groups. The interaction of the rest of the functors with products of semitopological groups remains unclear:

Problem 4.1. Which of the functors $T_3$, $\text{Reg}$, $\text{Tych}$ commute with products in the category of semitopological groups?

By Proposition 2.11, a paratopological group $G$ is feebly compact if and only if $\text{Reg}(G)$. Again, we do not know whether this result is valid for semitopological groups:

Problem 4.2. Is feebly compactness inverse invariant of the functors $T_1$, $T_2$, or $\text{Reg}$ in semitopological groups?
We recall that $X$ is called an $Oz$-space (equivalently, perfectly $\kappa$-normal) if every regular closed set in $X$ is a $G_\delta$-set. Evidently, $Oz$-spaces are Moscow. Notice that $Oz$-spaces are not assumed to satisfy the $T_0$ separation axiom.

**Problem 4.3.** Do the functors $T_2$ and/or $\text{Reg}$ respect the properties of being a Moscow and/or $Oz$-space?

It is not difficult to verify that, for a paratopological group $G$, if $\text{Reg}(G)$ is Moscow (or an $Oz$-space), then so is $G$. The same conclusion is also valid for the functor $T_2$. So we actually ask in Problem 4.3 whether the inverse implications hold.

Very recently I. Sánchez [14] answered Problem 4.2 in the affirmative for the functors $T_1$ and $T_2$.

**References**