Splittability over some classes of Corson compact spaces

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We study splittability over some classes $\mathcal{P}$ of compact spaces commonly used in functional analysis. We show that, for some nice classes $\mathcal{P}$, a compact space $X$ is splittable over $\mathcal{P}$ if and only if every function $f \in \mathbb{R}^X$ is reachable from $C_p(X)$ by a set belonging to $\mathcal{P}$. We also establish that every weakly Corson compact scattered space is Eberlein compact answering a question from [10]. We also prove that under $V = L$, a compact space $X$ is splittable over the class of Eberlein (Gul’ko, Corson) compact spaces if and only if $X$ is Eberlein (Gul’ko, Corson) compact.

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1. Introduction

The paper [11] introduced the class of spaces $X$ such that every $f \in \mathbb{R}^X$ is in the closure of a countable subset of $C_p(X)$; it was shown in [11] that this class has quite a few nice properties. Arhangel’skii and Shakhmatov proved in [6] that a space $X$ belongs to the above class if and only if for any $A \subset X$, there exists a continuous map $f : X \to \mathbb{R}^\omega$ such that $A = f^{-1}f(A)$ so it is natural to call such spaces splittable.

The idea of splitting a space along an arbitrary set was further developed by Arhangel’skii and other authors: Arhangel’skii called a space splittable over a class $\mathcal{P}$ if for any $A \subset X$ there exists a map $f : X \to Y$ such that $Y \in \mathcal{P}$ and $A = f^{-1}f(A)$. This concept turned out to be very fruitful and gave rise to an elegant theory of splittability (also called cleavability) over nice classes.

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Since splittability of a space $X$ over a class $\mathcal{P}$ is a consequence of having a weaker topology belonging to $\mathcal{P}$, many results on splittable space generalize well-known theorems on condensations. For example, if a pseudocompact space $X$ condenses onto a second countable one, then $X$ is compact and metrizable; Arhangel’skii and Shakhmatov established (see [6]) that the same conclusion is true if a pseudocompact space $X$ is splittable. Another result of [6] states that a Lindelöf space $X$ is splittable if and only if it has a weaker second countable topology.

Since every condensation of a compact space is a homeomorphism, it is especially interesting to consider splittability of a compact space $X$ over a class $\mathcal{P}$ and see how close to belonging to $\mathcal{P}$ the space $X$ should be. The results here are numerous so we will cite only a small portion of the most bright ones. It was proved in [3] that, for any cardinal $\kappa$, a compact space is splittable over the class of spaces of cardinality $\leq \kappa$ if and only if $|X| \leq \kappa$. If a compact space $X$ splits over the class of spaces of countable tightness (character) then $t(X) \leq \omega$ (or $\chi(X) \leq \omega$ respectively). Besides, if a compact space $X$ is splittable over $\mathbb{R}$, then $X$ embeds in $\mathbb{R}$. For the case $n > 1$, it is known that if compact space $X$ is splittable over $\mathbb{R}^n$, then it embeds in $\mathbb{R}^{n+1}$ (see Theorem 6.26 of [5]).

It turned out that splittability over some classes can also be approached by considering approximations of functions from $\mathbb{R}^X$. Call a space $X$ weakly splittable [10] if for every $f \in \mathbb{R}^X$ there exists a $\sigma$-compact subspace $Y \subset C_p(X)$ such that $f \in \overline{Y}$ (the closure is taken in the space $\mathbb{R}^X$). Since splittability is equivalent to reaching every $f \in \mathbb{R}^X$ from $C_p(X)$ by a countable set, every splittable space is weakly splittable. In [10] weakly splittable compact spaces were called weakly Eberlein compact. Since Eberlein compactness is characterized by existence of a dense $\sigma$-compact subspace of $C_p(X)$, it is immediate that every Eberlein compact is weakly Eberlein.

It was proved in [10], among other things, that a space $X$ is weakly splittable if and only if it is splittable over the class of Eberlein–Grothendieck spaces so we can see that approaching arbitrary functions on $X$ from $C_p(X)$ leads to the concept of splittability over a class. In this paper we exploit this idea in a more general context, i.e., assuming that every function $f \in \mathbb{R}^X$ is in the closure of a set $Y \subset C_p(X)$ from a class $\mathcal{P}$, we observe that this implies splittability of $X$ over a class $\mathcal{Q}$ and we find conditions on the space $X$ which imply that $C_p(X)$ has a dense subspace from the class $\mathcal{P}$.

We show that a scattered compact space splittable over the class of Corson compact spaces must be Eberlein compact. This solves Problem 4.8 from the paper [10]. We also establish that, under the Axiom of Constructibility, if a compact space $X$ is splittable over the class of Corson (Gu\’ko, Eberlein) compact spaces, then $X$ is Corson (Gu\’ko, Eberlein) compact. This shows that a positive answer to several other problems from [10] is consistent with ZFC.

2. Notation and terminology

All spaces are assumed to be Tychonoff; if $X$ is a space, then $\tau(X)$ is its topology. We denote by $\mathbb{R}$ the real line with its natural topology, $\mathbb{Q} \subset \mathbb{R}$ is the set of rationals and $\mathbb{D} = \{0, 1\} \subset \mathbb{R}$. As usual, $\omega$ is the set of all natural numbers and $\mathbb{N} = \omega \setminus \{0\}$. A space $X$ is scattered if every non-empty subspace $Y \subset X$ has an isolated point. Given a cardinal $\kappa > \omega$ we denote by $L(\kappa)$ the one-point Lindelöfication of a discrete space of cardinality $\kappa$. A space $X$ is called primarily Lindelöf if $X$ is a continuous image of a closed subset of $(L(\kappa))^\omega$ for some cardinal $\kappa$.

If $X$ is a space, then $C_p(X)$ is the set of continuous real-valued functions on $X$ endowed with the topology of pointwise convergence. Given a set $F \subset C_p(X)$ let $e_F(x)(f) = f(x)$ for any $x \in X$ and $f \in F$; then $e_F(x) \in C_p(F)$ for any $x \in X$ and $e_F : X \to C_p(F)$ is called the evaluation map with respect to $F$. If we have a continuous onto map $\varphi : X \to Y$, then $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(Y)$ and the map $\varphi^* : C_p(Y) \to C_p(X)$ is an embedding; it is called the dual map of $\varphi$.

A map $f : X \to Y$ is called condensation if it is a continuous bijection; in this case we say that $X$ condenses onto $Y$. Say that a family $\mathcal{F}$ of subsets of a space $X$ is a network modulo a cover $\mathcal{C}$ if for any
$C \in \mathcal{C}$ and $U \in \tau(X)$ with $C \subseteq U$ there exists $F \in \mathcal{F}$ such that $C \subseteq F \subseteq U$. A space $X$ is Lindelöf $\Sigma$ (or has the Lindelöf $\Sigma$-property) if there exists a countable family $\mathcal{F}$ of subsets of $X$ such that $\mathcal{F}$ is a network modulo a compact cover $\mathcal{C}$ of the space $X$. A space $X$ is Fréchet–Urysohn, if for any $A \subset X$ and $x \in \overline{A}$ there exists a sequence $\{a_n : n \in \omega\} \subset A$ such that $a_n \to x$.

Given an uncountable set $A$ the space $\Sigma(A) = \{x \in \mathbb{R}^2 : |x^{-1}(\mathbb{R}\setminus\{0\})| \leq \omega\}$ is called the $\Sigma$-product of real lines. A compact space $X$ is called Corson compact if $X$ embeds in $\Sigma(A)$ for some set $A$; the space $X$ is Gul’ko compact if $C_p(X)$ has the Lindelöf $\Sigma$-property and if $C_p(X)$ has a dense $\sigma$-compact subspace then $X$ is called Eberlein compact. Also, a compact space $X$ is Talagrand compact if $C_p(X)$ is $K$-analytic.

If $\mathcal{P}$ is a class of spaces, we say that a space $X$ is splittable over the class $\mathcal{P}$ if for any $A \subset X$ there exists a continuous map $f : X \to Y$ such that $Y \in \mathcal{P}$ and $A = f^{-1}(f(A))$. An essential percentage of authors use the term “cleavable” instead of “splittable”. A compact space $X$ is called weakly Corson (Gul’ko, Eberlein) compact if $X$ is splittable over the class of Corson (Gul’ko, Eberlein) compacta.

The rest of our terminology is standard and follows [8]; the survey of Hodel [9] can be consulted for definitions of cardinal invariants. We also use without reference the notation and the basic facts from the books [4] and [12] when dealing with $C_p$-theory.

3. Approximation of functions and splittability

We will consider the classes of compact spaces splittable over the class of Eberlein, Gul’ko and Corson compacta showing that in many cases, splittability of $X$ over a class $\mathcal{P}$ implies that $X$ belongs to $\mathcal{P}$.

### 3.1. Proposition

Given a class $\mathcal{P}$ invariant under countable products, suppose that $X$ is splittable over $\mathcal{P}$ and $f : X \to Y$ is a continuous map such that $Y \in \mathcal{P}$ and $f^{-1}(y) = \bigcup\{F_n^y : n \in \omega\}$ for every point $y \in Y$. Then there exists a space $Z \in \mathcal{P}$ and a continuous map $g : X \to Z$ such that $g^{-1}(g(F_n^y)) = F_n^y$ for any $y \in Y$ and $n \in \omega$.

**Proof.** By splittability of $X$ over $\mathcal{P}$, for the set $A_n = \bigcup\{F_n^y : y \in Y\}$ we can find a continuous map $f_n : X \to Z_n$ such that $Z_n \in \mathcal{P}$ and $f_n^{-1}f_n(A_n) = A_n$ for every $n \in \omega$. If $g$ is the diagonal product of the family $\{f_n : n \in \omega\}$ then $g : X \to Z = Y \times \prod\{Z_n : n \in \omega\}$ and $Z \in \mathcal{P}$. If $y \in Y$ and $x \in F_n^y$ for some $n \in \omega$ then $g(x') = g(x)$ implies that $x' \in f^{-1}(y)$ because $f(x) = f(x')$ so it follows from $f_n(x') = f_n(x)$ that $x' \in f^{-1}(y) \cap A_n = F_n^y$; therefore $g^{-1}g(x) \subseteq F_n^y$ for any $x \in F_n^y$. \(\square\)

### 3.2. Corollary

If a class $\mathcal{P}$ is invariant under countable products, suppose that $X$ is splittable over $\mathcal{P}$ and $f : X \to Y$ is a continuous map such that $Y \in \mathcal{P}$ and $|f^{-1}(y)| \leq \omega$ for every point $y \in Y$. Then $X$ condenses into a space belonging to the class $\mathcal{P}$.

**Proof.** If $f^{-1}(y) = \bigcup\{F_n^y : n \in \omega\}$ where every $F_n^y$ is a singleton, then Proposition 3.1 is applicable to conclude that there exists a continuous map $g : X \to Z$ such that $Z \in \mathcal{P}$ and, for each $z \in Z$, we have the inclusion $g^{-1}(z) \subset F_n^y$ for some $n \in \omega$ and $y \in Y$. Thus every $g^{-1}(z)$ has at most one element, so $g$ is injective. \(\square\)

Given a scattered compact space $X$ let $X_0 = X$; if $\beta$ is an ordinal and we have $X_\beta$ then $X_{\beta+1}$ is the set of non-isolated points of $X$. If $\alpha$ is a limit ordinal and we have $X_\beta$ for all $\beta < \alpha$ then $X_\alpha = \bigcap\{X_\beta : \beta < \alpha\}$. Since $X$ is scattered, the ordinal $\delta(X) = \min\{\alpha : X_\alpha = \emptyset\}$ is well-defined; it is called the dispersion index of $X$. It is easy to see that the dispersion index of $X$ is a successor ordinal for any non-empty scattered compact space $X$.

Alster established in [1] that every scattered Corson compact space is Eberlein compact. The following theorem generalizes this result.
3.3. Theorem. Suppose that $X$ is a scattered compact space. If $X$ is splittable over the class of Corson compact spaces, then $X$ is Eberlein compact.

Proof. We will apply induction on the dispersion index of $X$. If $di(X) = 1$ then $X$ is finite so there is nothing to prove. Now assume that $di(X) = \alpha$ and every scattered compact space $Y$ splittable over the class of Corson compact spaces and satisfying the condition $di(Y) = \beta < \alpha$ must be Eberlein compact. There is an ordinal $\mu$ such that $\alpha = \mu + 1$; evidently, the set $\mathcal{X}_\mu$ is finite. Choose a continuous map $f : X \to K$ such that $K$ is Corson compact and $f^{-1}(X_\mu) = X_\mu$; this implies that $f^{-1}(y) \subset X \setminus X_\mu$ for any $y \in K \setminus f(X_\mu)$. It easily follows from compactness of $f^{-1}(y)$ that $di(f^{-1}(y)) < \alpha$ so we can apply the induction hypothesis to convince ourselves that $f^{-1}(y)$ is Corson compact for any $y \in K \setminus f(X_\mu)$. If $y \in f(X_\mu)$ then $f^{-1}(y) \subset X_\mu$ is finite so it is a Corson compact space as well.

Applying Corollary 1 of [1] we can see that $f^{-1}(y)$ is strong Eberlein compact and hence it must be $\sigma$-discrete for any $y \in K$. Choose a family $\{F^n : n \in \omega\}$ of discrete subsets of $X$ such that $f^{-1}(y) = \bigcup\{F^n : n \in \omega\}$ for any $y \in K$. By Proposition 3.1, there is a continuous map $g : X \to L$ of $X$ onto a Corson compact space $L$ such that, for any $z \in L$, there exist $n \in \omega$ and $y \in K$ such that $g^{-1}(z) \subset F^n$; the set $F^n$ being discrete, the compact subspace $g^{-1}(z)$ must be finite for every $z \in L$. Finally, apply Corollary 3.2 to see that there exists a condensation (which is automatically a homeomorphism) of $X$ into a Corson compact space so $X$ is Corson compact and hence Eberlein compact by [1, Corollary 1].

The following corollary answers Problem 4.8 from the paper [10].

3.4. Corollary. Suppose that $P \in \{Gul’ko compact spaces, Talagrand compact spaces, Eberlein compact spaces\}$. If a scattered compact space $X$ is splittable over the class $P$, then $X$ is Eberlein compact.

Call a class $P$ complete if $P$ is invariant under closed subspaces, continuous images, every compact space belongs to $P$ and any countable product as well any countable union of spaces from $P$ belongs to $P$. Furthermore, we will denote by $\mathcal{F}(P)$ the class of spaces $X$ which embed into $C_p(Y)$ for some $Y \in P$.

Jardón proved in [10] that every $f \in \mathbb{R}^X$ is in the closure of a $\sigma$-compact subset of $C_p(X)$ if and only if $X$ is splittable over the class of Eberlein–Grothendieck spaces. The following result shows that this statement is true in a much wider context.

3.5. Theorem. If $P$ is a complete class of spaces, then the following conditions are equivalent:

(a) For any $f \in \mathbb{R}^X$, there exists a set $P \subset C_p(X)$ such that $P \in P$ and $f \in P$ (the bar denotes the closure in $\mathbb{R}^X$);

(b) the space $X$ is splittable over the class $\mathcal{F}(P)$.

Proof. If (a) holds then fix an arbitrary set $A \subset X$ and let $f(x) = 1$ for every $x \in A$ and $f(x) = 0$ if $x \notin A$. There exists a set $Y \subset C_p(X)$ such that $f \in \overline{Y}$ and $Y \in P$. Denote by $\varphi : X \to C_p(Y)$ the evaluation map, i.e., $\varphi(x)(f) = f(x)$ for any $f \in Y$. The space $Z = \varphi(X) \subset C_p(Y)$ belongs to the class $\mathcal{F}(P)$. If $x \in A$ and $y \notin A$ then $f(x) = 1$ and $f(y) = 0$ so it follows from $f \in \overline{Y}$ that there exists $g \in Y$ such that $g(x) > \frac{1}{2}$ and $g(y) < \frac{1}{2}$. In particular, $g(x) \neq g(y)$ which implies that $\varphi(x) \neq \varphi(y)$. As a consequence, $\varphi(A) \cap \varphi(X \setminus A) = \emptyset$ so $A = \varphi^{-1}\varphi(A)$, i.e., $X$ is splittable over the class $\mathcal{F}(P)$.

To prove the implication (b) $\implies$ (a) consider first any function $f \in \mathbb{D}^X$ and let $A = f^{-1}(1)$. By (b), there exists a space $P \in P$ and a continuous onto map $\varphi : X \to Y \subset C_p(P)$ such that $\varphi^{-1}(A) = A$. If $e : P \to C_p(Y)$ is the evaluation map then $Q = e(P) \subset C_p(Y)$ while $Q$ separates the points and the closed subsets of $Y$ and $Q \in P$. It is an easy consequence of the theorem of Stone–Weierstrass that the algebra $E$ generated by $Q$ in $C_p(Y)$ is dense in $C_p(Y)$. Since $E$ is the countable union of continuous images of spaces $Q^n \times \mathbb{R}^n$ for some $n \in \mathbb{N}$, completeness of the class $P$ implies that $E \in P$. 
The dual map \( \varphi^* : \mathbb{R}^Y \to \mathbb{R}^X \) defined by \( \varphi^*(h) = h \circ \varphi \) for any \( h \in \mathbb{R}^Y \) is an embedding so the set \( D = \varphi^*(E) \) also belongs to \( \mathcal{P} \); since \( \varphi^*(C_p(Y)) \subseteq C_p(X) \), we have \( D \subseteq C_p(X) \). Let \( g(y) = 1 \) for any \( y \in \varphi(A) \) and \( g(y) = 0 \) whenever \( y \in Y \setminus \varphi(A) = \varphi(X \setminus A) \). It is immediate that \( f = \varphi^*(g) \) so it follows from \( g \in \mathcal{E} \) that \( f \in \mathcal{D} \). Therefore we proved that

(1) for any \( f \in \mathcal{D}^X \) there exists a set \( P \subseteq C_p(X) \) such that \( P \in \mathcal{P} \) and \( f \in \mathcal{P} \).

Now, if \( f \) is an arbitrary element of \( \mathbb{R}^X \), then there exists a countable set \( N \subseteq \{q_1f_1 + \ldots + q_nf_n : n \in \mathbb{N}, q_i \in \mathbb{Q} \} \) such that \( f \in \overline{N} \). By (1), every function \( q_1f_1 \) is in the closure of a set \( P_i \subseteq C_p(X) \) with \( P_i \in \mathcal{P} \). Now it follows from completeness of \( \mathcal{P} \) and continuity of summation in \( C_p(X) \) that a continuous image of \( P_1 \times \ldots \times P_n \) lying in \( C_p(X) \) contains the function \( q_1f_1 + \ldots + q_nf_n \) in its closure. Thus, for every \( h \in N \) there exists a set \( Q_h \in \mathcal{P} \) such that \( Q_h \subseteq C_p(X) \) and \( h \in Q_h \). Therefore \( Q = \bigcup \{Q_h : h \in N\} \) is a subspace of \( C_p(X) \) which belongs to \( \mathcal{P} \) and contains \( f \) in its closure, i.e., (a) holds.

3.6. Corollary. Suppose that \( \mathcal{P} \) is a complete class of spaces and \( X \) is splittable over \( F(\mathcal{P}) \). If \( |X| \leq \mathfrak{c} \), then \( C_p(X) \) has a dense subspace \( Q \in \mathcal{P} \) and therefore \( X \) condenses onto a space from \( F(\mathcal{P}) \).

Proof. It follows from \( |X| \leq \mathfrak{c} \) that \( \mathbb{R}^X \) is separable; pick a countable dense set \( E \subseteq \mathbb{R}^X \). For every \( f \in E \), Theorem 3.5 implies that there exists a set \( Q_f \subseteq \mathcal{P} \) such that \( Q_f \subseteq C_p(X) \) and \( f \in \overline{Q_f} \). The set \( Q = \bigcup \{Q_f : f \in E\} \) belongs to \( \mathcal{P} \) and \( f \in \overline{Q} \) for any \( f \in E \) so \( Q \) is dense in \( C_p(X) \). Therefore the evaluation map \( \varphi : X \to C_p(Q) \) must be injective (see Problem 166 of [12]).

3.7. Corollary. If \( X \) is a compact space which splits over the class of Corson (Gul’ko) compact spaces and \( |X| \leq \mathfrak{c} \) then \( X \) is Corson compact (or Gul’ko compact respectively).

Proof. The class \( \mathcal{P} \) of primarily Lindelöf spaces (Lindelöf \( \Sigma \)-spaces) is complete and a compact space \( Z \) is Corson (Gul’ko) compact if and only if \( Z \) embeds into \( C_p(Y) \) for some \( Y \in \mathcal{P} \), i.e., our space \( X \) is splittable over the class \( F(\mathcal{P}) \). Corollary 3.6 shows that \( X \) condenses onto a space from \( F(\mathcal{P}) \), i.e., \( X \) is homeomorphic to a space from \( F(\mathcal{P}) \) which proves that \( X \) is Corson compact (or Gul’ko compact respectively).

3.8. Theorem. Assume that we are given a class \( \mathcal{P} \in \{ \text{Corson compact spaces, Gul’ko compact spaces, Talagrand compact spaces, Eberlein compact spaces} \} \). If a compact space \( X \) is splittable over \( \mathcal{P} \) and there exists a map \( f : X \to Y \in \mathcal{P} \) such that \( f^{-1}(y) \) is \( \sigma \)-metrizable for any \( y \in Y \), then \( X \in \mathcal{P} \).

Proof. Choose a family \( \{F^n_y : n \in \omega\} \) of metrizable subsets of \( Y \) such that \( f^{-1}(y) = \bigcup \{F^n_y : n \in \omega\} \) for any \( y \in Y \). By Proposition 3.1, there is a continuous map \( g : X \to L \) of \( X \) onto a compact space \( L \in \mathcal{P} \) such that, for any \( z \in L \), there exist \( n \in \omega \) and \( y \in K \) such that \( g^{-1}(z) \subseteq F^n_y \); the set \( F^n_y \) being metrizable, the compact subspace \( g^{-1}(z) \) is metrizable as well so we can find sets \( G^0_z \) and \( G^1_z \) such that \( g^{-1}(z) = G^0_z \cup G^1_z \) and every compact set \( K \subseteq G^i_z \) is countable for any \( i \in \{0, 1\} \) (see Corollary 2.5 of [7]). Now, apply Proposition 3.1 once again to find a continuous onto map \( h : X \to M \in \mathcal{P} \) such that for any \( y \in M \), there exist \( z \in L \) and \( i \in \{0, 1\} \) such that \( h^{-1}(y) \subseteq G^i_z \); thus, \( h^{-1}(y) \) is countable for any \( y \in M \). Finally, observe that Corollary 3.2 implies that there exists a condensation of \( X \) onto a space \( Z \in \mathcal{P} \); therefore \( X \) is homeomorphic to \( Z \) and hence \( X \in \mathcal{P} \).

3.9. Corollary. Assume that we are given a class \( \mathcal{P} \in \{ \text{Corson compact spaces, Gul’ko compact spaces, Talagrand compact spaces, Eberlein compact spaces} \} \). If a compact \( \sigma \)-metrizable space \( X \) is splittable over \( \mathcal{P} \), then \( X \in \mathcal{P} \).
Given a cardinal $\mu$, let $s_0(\mu) = \mu$; if $\alpha$ is an ordinal and we have $s_\alpha(\mu)$, then $s_{\alpha+1}(\mu) = (s_\alpha(\mu))^+$. If $\alpha$ is a limit ordinal and we have $s_\beta(\mu)$ for any $\beta < \alpha$, then $s_\alpha(\mu) = \sup\{s_\beta(\mu) : \beta < \alpha\}$. Let us consider the following statement:

(ACP) for every cardinal $\mu \geq \kappa$, there is a cardinal $\alpha < \kappa$ such that $\mu^\alpha \leq s_\alpha(\mu)$.

The assumption ACP was introduced in the paper [7]. In the same paper, the statement ACP&($\omega_1 < \kappa$) was denoted by ACP#. It is well known that ACP# is consistent with ZFC. We will need the following result of Bregman, Šapirovskij and Šostak (see Corollaries 8.9 and 9.7 of the paper [7]).

3.10. Theorem. If either ACP# or the Axiom of Constructibility ($V = L$) is assumed, then for any Hausdorff space $X$, we can find sets $X_0, X_1 \subset X$ such that $X = X_0 \cup X_1$ and every compact $K \subset X_i$ is scattered for every $i \in \{0, 1\}$.

It was asked in the paper [10] whether splittability of a compact space over the class of Eberlein (Corson, Gul’ko) compact spaces must be an Eberlein (Corson, Gul’ko) compact space. We will show next that the positive answer to these questions is consistent with ZFC. As a consequence, we give a consistent positive answer to all open problems formulated in [10] except Problem 4.4. Recall that Corollary 3.4 answered Problem 4.8 in ZFC.

3.11. Theorem. Assume that either ACP# or $V = L$ holds and we are given a class $\mathcal{P} \in \{\text{Corson compact spaces, Gul’ko compact spaces, Talagrand compact spaces, Eberlein compact spaces}\}$. If a compact space $X$ is splittable over $\mathcal{P}$, then $X$ belongs to $\mathcal{P}$.

Proof. Apply Theorem 3.10 to find disjoint subsets $X_0$ and $X_1$ of the space $X$ such that $X = X_0 \cup X_1$ and every compact $K \subset X_i$ is scattered for every $i \in \{0, 1\}$. There exists a continuous map $f : X \to Y$ such that $Y \in \mathcal{P}$ and $f^{-1}(X_i) = X_i$ for each $i \in \{0, 1\}$. For any $y \in Y$, the set $f^{-1}(y)$ is compact and $f^{-1}(y) \subset X_i$ for some $i$; therefore it is scattered. Since splittability over the class $\mathcal{P}$ is a hereditary property, the set $f^{-1}(y)$ is splittable over $\mathcal{P}$ so it is an Eberlein compact by Theorem 3.3. Applying Corollary 1 of [1] we can see that $f^{-1}(y)$ is strong Eberlein compact so it must be $\sigma$-discrete and hence $\sigma$-metrizable for any $y \in Y$. Finally, apply Theorem 3.8 to conclude that the space $X$ has to belong to $\mathcal{P}$. \qed

Recall that a compact space $X$ is weakly Corson (Gul’ko) compact if $X$ splits over the class of Corson (Gul’ko) compacta. Suppose that $\mathcal{P} \in \{\text{Corson compact spaces, Gul’ko compact spaces}\}$. Since there is still a hope that compact spaces from weak class $\mathcal{P}$ might fail to be in $\mathcal{P}$, it is of interest to find out what properties of the class $\mathcal{P}$ the spaces from the weak class $\mathcal{P}$ have in ZFC. An extensive study of the properties of weak Eberlein compacta was undertaken in [10] and several methods used in [10] also apply to study weakly Corson/Gul’ko compacta.

3.12. Proposition. Suppose that $X$ is a weakly Corson compact space. Then

(a) $X$ is $\omega$-monolithic and Fréchet–Urysohn;
(b) if $A \subset X$ and $|A| \leq \kappa$, then $\overline{A}$ is Corson compact;
(c) if $Y$ is a continuous image of $X$ and $d(Y) \leq \kappa$, then $Y$ is Corson compact;
(d) the space $C_p(X)$ is Lindelöf.

Proof. Observe that Corson compact spaces are Fréchet–Urysohn so we can apply Corollary 18 of the paper [3] to see that any weakly Corson compact space is also Fréchet–Urysohn. If a compact space is splittable over a class of $\omega$-monolithic spaces, then it is $\omega$-monolithic by [3, Theorem 36], i.e., we proved the property (a).
Suppose that $X$ is weakly Corson compact and $A \subset X$ is a set of cardinality at most $c$. Fréchet–Urysohn property of $X$ implies that $|\overline{A}| \leq c$; it is evident that $\overline{A}$ is also weakly Corson compact so we can apply Corollary 3.7 to see that $\overline{A}$ has to be Corson compact. This completes the proof of (b).

To prove (c), assume that $X$ is weakly Corson compact and $f : X \to Y$ is a continuous onto map such that $d(Y) \leq c$. It is easy to find a set $A \subset X$ such that $|A| \leq c$ and $f(A)$ is dense in $Y$. Then $K = \overline{A}$ is Corson compact by (b) so $Y = f(K)$ is also Corson compact being a continuous image of $K$.

To see that (d) holds, assume that $C_p(X)$ is not Lindelöf; then $\text{ext}(C_p(X)) = l(C_p(X)) > \omega$ (see Theorem III.6.1 of [4]) so we can find a closed discrete set $D \subset C_p(X)$ with $|D| = \omega_1$. The evaluation map $\varphi : X \to C_p(D)$ is continuous and $w(Y) \leq \omega_1$ where $Y = \varphi(X)$. Observe that the dual map $\varphi^* : C_p(Y) \to C_p(X)$ embeds $C_p(Y)$ in $C_p(X)$ as a closed subspace and $D \subset \varphi^*(C_p(Y))$. Therefore $\text{ext}(C_p(Y)) > \omega$ and hence $C_p(Y)$ is not Lindelöf while $Y$ is Corson compact by (c) which is a contradiction.

\[\square\]

3.13. Theorem. If $X$ is a weakly Gul’ko compact space, then $w(X) = c(X)$.

Proof. Let $\kappa = \max\{c(X), c\}$ and take any Lindelöf $\Sigma$-space $Y \subset C_p(X)$. It is easy to find a family $\mathcal{F}$ of compact subsets of $Y$ such that $Y = \bigcup \mathcal{F}$ and $|\mathcal{F}| \leq c$. It follows from Theorem III.5.9 of [4] that $w(K) \leq c(X) \leq \kappa$ for any $K \in \mathcal{F}$. Therefore $nw(Y) = nw(\bigcup \mathcal{F}) \leq \kappa \cdot c = \kappa$ so we established that

(*) if $Y \subset C_p(X)$ is a Lindelöf $\Sigma$-space, then $nw(Y) \leq \kappa$ and hence $d(Y) \leq \kappa$.

We will prove first that $\chi(X) \leq \kappa$. Fix a point $x \in X$; letting $f(x) = 1$ and $f(y) = 0$ for any $y \in X\setminus\{x\}$, we obtain a function $f \in \mathbb{R}_X$. Apply the weak Gul’ko property to find a Lindelöf $\Sigma$-space $Y \subset C_p(X)$ such that $f \in Y$ (the bar denotes the closure in $\mathbb{R}_X$). By (*) we can find a set $D \subset Y$ such that $Y \subset \overline{D}$ and $|D| \leq \kappa$.

Observe that $Q_g = g^{-1}(g(x))$ is a $G_\delta$-set for any $g \in D$ so $Q = \bigcap\{Q_g : g \in D\}$ is a $G_\kappa$-subset of $X$ and $x \in Q$. If $y \in X\setminus\{x\}$, then it follows from $f \in \overline{Y} \subset \overline{D}$ that we can find a function $g \in D$ such that $g(x) > \frac{1}{2}$ and $g(y) < \frac{1}{2}$ and, in particular, $g(x) \neq g(y)$, i.e., $y \notin Q_g$. This shows that $Q = \{x\}$ and hence $\{x\}$ is a $G_\kappa$-set in $X$ for any $x \in X$. By compactness of $X$ we have $\chi(X) \leq \kappa$.

Applying compactness of the space $X$ again, we conclude that $|X| \leq 2^\kappa$ (see [8, Theorem 3.1.29]) and hence $d(\mathbb{R}_X) \leq \kappa$; pick a dense set $G \subset \mathbb{R}_X$ such that $|G| \leq \kappa$. For every $g \in G$ there exists a Lindelöf $\Sigma$-space $L_g \subset C_p(X)$ such that $g \in L_g$; it is immediate that $L = \bigcup\{L_g : g \in G\} \subset C_p(X)$ is dense in the space $\mathbb{R}_X$ and hence in $C_p(X)$. Apply the property (*) again to convince ourselves that we have the inequalities $nw(L) \leq \sup\{nw(L_g) : g \in G\} \leq \kappa \cdot \kappa = \kappa$. As an immediate consequence, $d(C_p(X)) \leq d(L) \leq nw(L) \leq \kappa$ which shows that $w(X) = nw(X) = d(C_p(X)) \leq \kappa$ (see Problem 174 of [12]) so $w(X) \leq \kappa$. If $w(X) \leq c$ then $d(X) \leq \kappa$ and hence $|X| \leq c$ by Proposition 3.12(a) so $X$ is Gul’ko compact by Corollary 3.7. The weight of any Gul’ko compactum is equal to its Souslin number (see [2]) so $w(X) = c(X)$. The case of $w(X) \leq c(X)$ is trivial so we have $w(X) = c(X)$ for every weakly Gul’ko compact space $X$. \[\square\]

4. Open problems

Since it is consistent with the usual axioms of set theory that every weakly Eberlein compact space is Eberlein compact, the most intriguing question is whether weak Eberlein compacta which are not Eberlein exist in some models of ZFC. Besides, it is important to find out what properties of weak Eberlein compacta are provable without any additional set-theoretic axioms.

4.1. Question. Suppose that a compact space $X$ is splittable over the class of Corson compact spaces. Is it true in ZFC that $X$ is Corson compact?
4.2. Question. Suppose that a compact space $X$ is splittable over the class of Corson compact spaces. Is it true in ZFC that $w(X) = d(X)$?

4.3. Question. Suppose that a compact space $X$ is splittable over the class of Corson compact spaces. Is it true in ZFC that any continuous image of $X$ is also splittable over the class of Corson compacta?

4.4. Question. Suppose that a compact space $X$ is splittable over the class of Corson compact spaces. Is it true in ZFC that $X \times X$ is also splittable over the class of Corson compacta?

4.5. Question. Suppose that a compact space $X$ is splittable over the class of Gul’ko compact spaces. Is it true in ZFC that $X$ is Gul’ko compact?

4.6. Question. Suppose that a compact space $X$ is splittable over the class of Gul’ko compact spaces. Is it true in ZFC that any continuous image of $X$ is also splittable over the class of Gul’ko compacta?

4.7. Question. Suppose that a compact space $X$ is splittable over the class of Gul’ko compact spaces. Is it true in ZFC that $X \times X$ is also splittable over the class of Gul’ko compacta?

4.8. Question. Suppose that a compact space $X$ is splittable over the class of Gul’ko compact spaces. Is it true in ZFC that $X$ has a dense metrizable subspace?

4.9. Question. Suppose that a compact space $X$ is splittable over the class of Eberlein compact spaces. Is it true in ZFC that $X$ is Eberlein compact?

4.10. Question. Suppose that a compact space $X$ is splittable over the class of Eberlein compact spaces. Is it true in ZFC that $X$ has a dense metrizable subspace?

4.11. Question. Suppose that a compact space $X$ is splittable over the class of Eberlein compact spaces. Is it true in ZFC that any continuous image of $X$ is also splittable over the class of Eberlein compacta?

4.12. Question. Suppose that a compact space $X$ is splittable over the class of Eberlein compact spaces. Is it true in ZFC that $X \times X$ is also splittable over the class of Eberlein compacta?

4.13. Question. Suppose that $X$ is a compact space such that $\overline{A}$ is Corson compact for any $A \subset X$ with $|A| \leq c$. Must $X$ be Corson compact?

4.14. Question. Suppose that $X$ is a compact space such that $\overline{A}$ is Gul’ko compact for any $A \subset X$ with $|A| \leq c$. Must $X$ be Gul’ko compact?

4.15. Question. Suppose that $X$ is a compact space such that $\overline{A}$ is Eberlein compact for any $A \subset X$ with $|A| \leq c$. Must $X$ be Eberlein compact?

4.16. Question. Suppose that a pseudocompact space $X$ is splittable over the class of Eberlein compact spaces. Must $X$ be Eberlein compact?

4.17. Question. Suppose that a countably compact space $X$ is splittable over the class of Eberlein compact spaces. Must $X$ be Eberlein compact?

4.18. Question. Suppose that $X$ is a compact space such that $\overline{A}$ is a Lindelöf $\Sigma$-space for any $A \subset C_p(X)$ with $|A| \leq c$. Must $X$ be Gul’ko compact?
References