An Introduction to Extremal Kähler Metrics

Gábor Székelyhidi

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MAA REVIEW

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A Kähler manifold is a complex manifold with a Riemannian structure given by a closed Hermitian form, its Kähler metric, or, equivalently, with a symplectic structure compatible with the integral complex structure. Examples of Kähler manifolds are the complex Euclidean space $\mathbb{C}^n$ with its standard Hermitian metric and complex projective space $\mathbb{C}P^n$, equipped with the Fubini-Study metric. Since any submanifold of a Kähler manifold is obviously Kähler, all smooth complex projective algebraic varieties are Kähler and so are Stein manifolds. Moreover, Kähler manifolds with a positive line bundle can be embedded into projective space by a well-known theorem of Kodaira's.

An important subfamily of Kähler manifolds is given by the Kähler-Einstein manifolds, i.e., those Kähler manifolds whose Ricci tensor is a scalar multiple of their metric tensor. Questions on which Kähler manifolds are Kähler-Einstein, precisely formulated, are part of the so-called Calabi conjectures. One important case of these conjectures, solved by S. T. Yau in the late 1970s, established as by-product the existence of a subfamily of Kähler manifolds, the Calabi-Yau manifolds, whose importance goes well beyond the realm of complex geometry. More refinements or related conjectures consider the case of compact Kähler manifolds and the sign of their first Chern class. For example, Yau conjectures that when the first Chern class is positive, a Kähler manifold is Kähler-Einstein if and only if it is K-stable, in the sense of Mumford’s geometric invariant theory.

As may be gathered from this sketchy outline, Kähler manifolds are at the junction of several branches of mathematics. Analysis and partial differential equations, as exemplified by Yau's
original work, are fundamental tools to understand the rich differential, symplectic, and complex algebraic geometry underlying these manifolds.

The book under review is devoted to study a class Kähler manifolds which come equipped with an extremal Kähler metric, that is, a metric given as a critical point of a natural energy functional, the $L^2$-norm of the curvature of the metric. Examples of these extremal metrics are the Kähler-Einstein metrics and constant scalar curvature Kähler metrics. These extremal metrics, introduced by Eugenio Calabi in the 1980s, are higher dimensional analogues of the canonical metric given by Riemann’s uniformization theorem in the one-dimensional case.

The book gives a fast-paced introduction to recent work in this field, particularly to the question on the existence of such extremal metrics. The first three chapters summarize the general analytic requirements, Kähler geometry, and Kähler-Einstein metrics. Extremal metrics are introduced in chapter four and chapter five summarizes the basic facts on geometric invariant theory and symplectic geometry, sketching the proof of the main result (the Kempf-Ness theorem) relating the quotients in both settings. Chapter seven discusses recent relative results of Donaldson and Fujiki on the curvature viewed as a moment map, clarifying some results first discussed on chapter four, and at the same time motivating the introduction of the K-stability condition as an analogue of Mumford’s stability condition in geometric invariant theory. Although most of the results are just sketched, carefully chosen examples such as ruled surfaces or toric varieties, help to clarify them. As emphasized by the author, the Bergman kernel, that is, analysis and PDEs, provide the crucial link between the algebraic geometry and differential geometry of these manifolds. All of these are discussed in chapter seven, including a proof of a theorem of Donaldson’s and as a consequence a characterization of K-stability for a constant scalar metric Kähler manifold. The last chapter treats recent work by Arezzo and Pacard on the construction of constant scalar metrics on the blow-up at a point of compact constant scalar metric Kähler manifold.

This is an important book, in a rapidly-developing area, that brings the specialist or graduate student working on Kähler geometry to the frontiers of today research. It is not a self-contained textbook. The pre-requisites in geometric invariant theory, for example, would require some devotion from a potential reader grounded on Riemannian geometry; vice-versa, a reader brought-up in algebraic geometry would have to make an effort to follow the part on analysis or differential geometry. The rewards for these efforts justify everything: the book is well organized and when it sketches an argument there are precise pointers to the literature for full details.

Felipe Zaldivar is Professor of Mathematics at the Universidad Autonoma Metropolitana-I, in Mexico City. His e-mail address is fz@xanum.uam.mx.

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