Over an algebraically closed field, a linear algebraic group is a smooth affine variety whose underlying set is an abstract group in such a way that the operation of the group and taking inverses are algebraic maps. These groups can be realized as groups of matrices, and as such are quite ubiquitous in mathematics.

Of course, many of these groups of matrices are sometimes defined and studied over non-algebraically closed fields, such as number fields, finite or local fields, or even the field of real numbers. In the middle of the twentieth century, when the study of these groups became important, algebraic geometry was going through a massive metamorphosis, from the just established Weil foundations to the emerging scheme-theoretical language of Grothendieck. Thus, many results on algebraic groups were originally formulated in the "old" Weil language. Even the standard textbooks, all with the same title *Linear Algebraic Groups*, of A. Borel (First Edition, Benjamin, 1969, Second Edition, Springer, 1991), T. Springer (Birkhäuser, First Edition 1981, Second Edition 1998, Reprinted in 2009), and J. Humphreys (Springer, 1975) chose to follow Weil’s *Foundations of Algebraic Geometry* (AMS, First Edition 1946, 10th printing with additions and corrections, 2000) for their geometric language, instead of Grothendieck’s *Éléments de Géométrie Algébrique* (Pub. Math. IHES, 1960-1967) and its language of group schemes.

For algebraically closed fields, the theory of algebraic groups in either language essentially yields the same results. The main classification results were obtained by Chevalley in the mid twentieth century. Broadly, the starting points are the following. Over an algebraically closed field, any square invertible matrix can be decomposed as a product of a semisimple matrix (diagonalizable) and a...
unipotent matrix (all eigenvalues equal to 1). This decomposition can be extended to an arbitrary algebraic affine algebraic group using Chevalley's theorem, which embeds the given group as a closed group of a linear group.

The set of unipotent elements is a closed subgroup of the given group, and a group is a unipotent group if it is equal to its unipotent subgroup. A reductive group over an algebraically closed field is an algebraic group whose maximal normal connected unipotent subgroup (its so-called unipotent radical) is trivial. The main reason for the interest on the structure of unipotent and reductive groups is that every algebraic group is an extension of a unipotent group by a reductive group. The classification and structure of reductive groups over an algebraically closed field was one of the crowning achievements obtained by Chevalley, essentially completing the classification program.

If one now considers an arbitrary field, the two languages of algebraic geometry differ, in particular for the study of algebraic groups. Grothendieck's foundations are more flexible, for example allowing the study algebraic groups over any base scheme intrinsically.

To treat algebraic groups $G$ over a non-algebraically closed field $k$, the usual method is to extend scalars to an algebraic closure $k^a$ of the given field $k$ and look at the corresponding base-changed group $G_{k^a}$. The geometry is then done on $G_{k^a}$, and to obtain corresponding results for the group $G$ over the given field $k$, one must try to descend the corresponding structures or results. Consider, for example, the unipotent radical. We have a definition of the unipotent radical of $G$ over $k^a$. If $k$ is a perfect field, this unipotent radical descends to $G$ over $k$, and $G$ is an extension of this unipotent radical over $k$ and the quotient of $G$ by this unipotent radical over $k$. Clearly, the unipotent radical of this quotient is trivial. Thus, over perfect fields, defining reductive groups over $k$ as groups $G$ whose base-changed group $G_{k^a}$ over an algebraic closure of $k$ are reductive works fine, essentially because descent theory works fine when the field extension $k^a/k$ is Galois. The classification of reductive groups over non-algebraically closed fields was obtained by A. Borel and J. Tits in mid 1960s, following the lines of the algebraically closed case and being careful on the rationality questions.

However, for fields that are not perfect the approach sketched above does not work. Nevertheless, one still would like to study and classify algebraic groups over imperfect fields. In that case, the objective is to study these groups intrinsically, over the given field $k$.

This is far from trivial, since there are some basic facts that seem natural and expected but are no longer true over non-algebraically closed imperfect fields. For example, over an algebraically closed field one can identify an affine algebraic group with its group of rational points over the given field. But for non-algebraically closed fields this is no longer true: the Zariski closure of the set of rational points could be a proper subset of the given group. Another such counter-intuitive fact is that there are algebraic groups defined over imperfect fields $k$ that do not have a nontrivial normal connected and unipotent subgroup defined over $k$, but that when base-changed to the group $G_{k^a}$ over an algebraic closure of $k$ are not reductive. These are the pseudo-reductive groups.

For algebraic groups over an arbitrary field, the definition of unipotent just asks that the corresponding subgroup be defined over the given field. Thus, a pseudo-reductive group, over $k$, is an algebraic group defined over $k$, such that its maximal normal connected unipotent $k$-subgroup (its so-called unipotent radical over $k$) is trivial. As in the case of algebraically closed fields, the main reasons for the interest on understanding the structure of pseudo-reductive groups is that every algebraic group over $k$ is an extension of its unipotent radical by the corresponding quotient of the group by its unipotent radical, and this quotient is clearly pseudo-reductive.
The monograph under review is devoted to the elucidation of the structure and classification of pseudo-reductive groups over imperfect fields, completing the program initiated by J. Tits, A. Borel and T. Springer in the last three decades of the last century. The main results include the fact that all pseudo-reductive groups over an imperfect field of characteristic different from 2 or 3 can be obtained using Weil restriction of scalars over a purely inseparable finite extension of the base field $k$ of a connected reductive group, direct products, or replacement of a Cartan subgroup by a certain commutative subgroup. This so-called standard construction is detailed in Part I of the monograph, including the structure theory of these groups and combinatorial data.

Part II is devoted to a deeper study of pseudo-reductive groups obtained by the standard construction, for example showing that the standard construction is invariant under replacement of the base field with a separable closure. Part III takes on the cases of characteristic 2 and 3, exhibiting a class of pseudo-reductive groups that only exist in these characteristics. The classification for characteristic 3 is complete, but the characteristic 2 case will only be completed in the forthcoming Princeton monograph by B. Conrad and G. Prasad.

There are three appendices, collecting some general basic facts on linear algebraic groups not usually found in the literature, e.g., Weil restriction, or giving complete proofs of some old results of Tits, just recently published in Volume IV of his Collected Works (Lectures on Algebraic Groups, pp. 657–740), or some rationality questions.

It should be clear now that this monograph is a remarkable achievement and the definitive reference for pseudo-reductive groups. It certainly belongs in the library of anyone interested on algebraic groups and their arithmetic and geometry.

This second edition has grown from 533 to 665 pages mainly to incorporate results on groups over imperfect fields of characteristic 2, whose main applications are definite classification theorems of pseudo-reductive groups, with no restriction on the field, to be included in the forthcoming Princeton monograph by Conrad and Prasad.

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