Monotone pseudobase assignments and Lindelöf $\Sigma$-property

V.V. Tkachuk

Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco, 186, Col. Vicentina, Iztapalapa, C.P. 09340, Mexico D.F., Mexico

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ABSTRACT

We introduce and study the spaces with $\kappa$-monotone pseudo-network (pseudobase) assignment. We show that the respective classes are invariant under arbitrary subspaces, countable products, and are lifted by condensations. Besides, the class of spaces with a $\kappa$-monotone pseudo-network assignment is preserved by $\sigma$-products. It is also proved that a countably compact space $X$ with an $\omega$-monotone pseudobase assignment is compact and metrizable. If a countably compact space $X$ has an $\omega$-monotone pseudo-network assignment, then $X$ is monotonically monolithic and hence Corson compact. In Lindelöf $\Sigma$-spaces, having a $\kappa$-monotone pseudo-network assignment is equivalent to being monotonically $\kappa$-monolithic. As an application of the above results in $C_p$-theory, we show that if $C_p(X)$ is a Lindelöf $\Sigma$-space and $s(X) = \omega$, then $X$ has a countable network; this solves an open problem published in 2001.

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1. Introduction

Recall that, for an infinite cardinal $\kappa$, a space $X$ is called $\kappa$-monolithic if $nW(\mathcal{A}) \leq \kappa$ for every set $A \subset X$ with $|A| \leq \kappa$. The space $X$ is monolithic if it is $\kappa$-monolithic for any infinite cardinal $\kappa$. These concepts were discovered by Arhangel’skii (see [2]), and turned out to be very useful both in the theory of cardinal invariants and $C_p$-theory.

In [13] Tkachuk introduced the class of monotonically monolithic spaces and proved that every subspace of a monotonically monolithic space has the $D$-property. In the same paper it is proved that $C_p(X)$ is monotonically monolithic for any Lindelöf $\Sigma$-space $X$. This implies that the class of monotonically monolithic

\[ E-mail address: vova@xanum.uam.mx. \]

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spaces is reasonably large; it was also established in [13] that monotone monolithity is invariant under countable products, subspaces and closed maps. In the paper [1] monotone $\kappa$-monolithity was introduced for any infinite cardinal $\kappa$; it was proved, among other things, that monotone $\kappa$-monolithity is preserved by countable products and $\sigma$-products.

Answering a question published in [1] Gruenhage established that every monotonically $\omega$-monolithic compact space must be Corson compact and gave an example of a Corson compact space that fails to be monotonically $\omega$-monolithic. In the paper [14] Tkachuk gave an example which shows that a monotonically monolithic compact space is not necessarily Gul’ko compact. This great variety of applications that popped up after monotonically monolithic spaces were introduced produced a stable interest in possible modifications and implications of monotone monolithity.

In particular, Peng defined in [9] the concept of weak monotone monolithy and introduced semi-monotonically monolithic spaces in [10]. He studied the general properties of these two notions and their relationship with the $D$-property.

In this paper we introduce the notions of a $\kappa$-monotone pseudobase assignment and $\kappa$-monotone pseudo-network assignment for any infinite cardinal $\kappa$; they are obtained by replacing bases with pseudobases and networks with pseudo-networks in the definition of monotone monolithy.

The spaces from these new classes need not be $\kappa$-monolithic but they still retain a lot of properties of monotonically monolithic spaces which implies that studying them we obtain more information about monotonically monolithic spaces and generalizations of some known results.

We prove, among other things, that the classes of spaces with a $\kappa$-monotone pseudo-network assignment are invariant under subspaces, countable products and $\sigma$-products; besides, they are inverse invariants for condensations. It is also proved that a countably compact space $X$ with an $\omega$-monotone pseudobase assignment is compact and metrizable. If a countably compact space $X$ has an $\omega$-monotone pseudo-network assignment, then $X$ is monotonically monolithic and hence Corson compact. In Lindelöf $\Sigma$-spaces, having a $\kappa$-monotone pseudo-network assignment is equivalent to being monotonically $\kappa$-monolithic. As an application of the above results in $C_p$-theory, we show that if $C_pC_p(X)$ is a Lindelöf $\Sigma$-space and $s(X) = \omega$, then $X$ is has a countable network; this gives a positive answer to an open problem published in 2001.

2. Notation and terminology

All spaces are assumed to be Tychonoff. Given a space $X$, the family $\tau(X)$ is its topology and $\mathcal{L}(X)$ is the family of all closed subsets of $X$. Furthermore, $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any point $x \in X$; given any set $A \subset X$ let $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. If $X$ is a set then $\exp(X) = \{Y : Y \subset X\}$; we will also need the subfamilies and $[X]^{<\kappa} = \{Y \in \exp(X) : |Y| < \kappa\}$ and $[X]^{\leq \kappa} = \{Y \in \exp(X) : |Y| \leq \kappa\}$ of $\exp(X)$ for any cardinal $\kappa$. As usual, $\mathbb{R}$ is the set of reals and $C_p(X)$ is the set of real-valued continuous functions on $X$ endowed with the pointwise convergence topology.

Say that a family $\mathcal{F}$ of subsets of a space $X$ is a network with respect to a cover $\mathcal{C}$ if for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$ there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$. If $\mathcal{C} = \{\{x\} : x \in X\}$ then a network with respect to $\mathcal{C}$ is called a network of $X$. The network weight $nw(X)$ of a space $X$ is the minimal cardinality of a network in $X$. A space that has a countable network is called cosmic.

A space $X$ is Lindelöf $\Sigma$ if there exists a countable family $\mathcal{F}$ of subsets of $X$ such that $\mathcal{F}$ is a network with respect to a compact cover $\mathcal{C}$ of the space $X$. If $X$ is a space then a family $\mathcal{G}$ of subsets of $X$ is called a network (base) at a point $x \in X$ if $\{G \subset \tau(X) \text{ and} \}$ for any $U \in \tau(x, X)$ there exists $G \in \mathcal{G}$ such that $x \in G \subset U$. Given a set $A$ in a space $X$ say that a family $\mathcal{N}$ of subsets of $X$ is an external network (base) of $A$ in $X$ if (all elements of $\mathcal{N}$ are open in $X$ and) $\mathcal{N}$ is a network at every $x \in A$.

For an infinite cardinal $\kappa$, say that a space $X$ is (strongly) monotonically $\kappa$-monolithic if, to any set $A \subset X$ with $|A| \leq \kappa$, we can assign an external network (base) $\mathcal{O}(A)$ of the set $\overline{A}$ in such a way that the following conditions are satisfied:
\( \{O(A) \mid |O(A)| \leq |A| \cdot \omega \) \]
(b) if \( A \subset B \) then \( O(A) \subset O(B) \);
(c) if \( \lambda \leq \kappa \) is an ordinal and we have a family \( \{A_\alpha : \alpha < \lambda \} \subset [X]^{\leq \kappa} \) such that \( \alpha < \beta < \lambda \) implies \( A_\alpha \subset A_\beta \), then \( O(\bigcup_{\alpha \leq \lambda} A_\alpha) = \bigcup_{\alpha < \lambda} O(A_\alpha) \).

A space \( X \) is (strongly) monotonically monolithic if it is (strongly) monotonically \( \kappa \)-monolithic for any infinite cardinal \( \kappa \).

Given a space \( X \) say that a family \( N \) is a pseudo-network (pseudobase) at a point \( x \in X \) if \( (N \subset \tau(X) \text{ and}) \) there exists a family \( \mathcal{N}' \subset N \text{ such that } \{x\} = \bigcap \mathcal{N}' \). The family \( N \) is a pseudo-network (pseudobase) in \( X \) if its is a pseudo-network (pseudobase) at every point of \( X \). If \( x \in X \), then \( \psi(x, X) \) is the minimal infinite cardinal \( \kappa \) such that there exists a pseudobase at \( x \) of cardinality \( \kappa \); besides, \( \psi(x, X) = \sup \{\psi(x, X) : x \in X\} \).

We use the Russian term condensation for a continuous bijection. A space \( X \) condenses onto a space \( Y \) if there exists a condensation \( f : X \to Y \). The cardinal \( \sup \{\{|D| : D \text{ is a discrete subspace of } X\} \cdot \omega \) is called the spread of \( X \) and is denoted by \( s(X) \).

If \( X \) is a space, then a map \( N : X \to \tau(X) \) is called a neighborhood assignment if \( x \in N(x) \) for any \( x \in X \). Say that \( X \) is a D-space if for every neighborhood assignment \( N : X \to \tau(X) \), there exists a closed discrete set \( D \subset X \) such that \( \bigcup\{N(x) : x \in D\} = X \).

The unexplained topological notions can be found in the book \([4] \); the survey of Hodel \([8] \) covers all that is necessary for the use of cardinal invariants. The book \([15] \) can be consulted to find the relevant facts and definitions of \( C_p \)-theory.

3. Pseudo-network and pseudobase assignments

A monotone assignment of external networks leads to the concept of monotone monolithicity. It turns out that monotone assignments of pseudo-networks and pseudobases give new interesting concepts with quite a few applications.

3.1. Definition. Suppose that \( \kappa \) is an infinite cardinal and we have sets \( X \) and \( Y \). Given a family \( A \subset \exp(X) \), a family \( B \subset \exp(Y) \) and a map \( \varphi : A \to B \), say that \( \varphi \) is \( \kappa \)-monotone if

\( (a) \ |\varphi(A)| \leq |A| \cdot \omega \) whenever \( A \in A \) and \( |A| \leq \kappa; \)
\( (b) \text{ if } A, B \in A \text{ and } A \subset B \text{ then } \varphi(A) \subset \varphi(B); \)
\( (c) \text{ if } \lambda \leq \kappa \text{ is a cardinal, } \{A_\alpha : \alpha < \lambda \} \subset A \text{ is a family such that } A_\alpha \subset A_\beta \text{ whenever } \alpha < \beta \text{ and } \)
\( A = \bigcup_{\alpha < \lambda} A_\alpha \in A, \text{ then } \varphi(A) = \bigcup_{\alpha < \lambda} \varphi(A_\alpha). \)

3.2. Lemma. Given sets \( X, Y \) and an infinite cardinal \( \kappa \), suppose that we have families \( A \subset \exp(X) \) and \( B \subset \exp(Y) \) such that \( \exp(A) \subset A \) for any \( A \in A \). Suppose that \( O : A \to B \) is a \( \kappa \)-monotone operator and consider the set \( G(A) = \bigcup\{O(H) : H \in [A]^{\leq \omega}\} \) for every \( A \in A \). Then \( G(A) \) coincides with \( O(A) \) for every \( A \in A \) with \( |A| \leq \kappa \).

Proof. It is immediate that \( G(A) \subset O(A) \) for any \( A \in A \) with \( |A| \leq \kappa \) so we only need to show that \( O(A) \subset G(A) \) for every such \( A \). Proceeding by transfinite induction assume first that \( |A| \leq \omega \); then \( A \) can be represented as \( \bigcup\{H_n : n \in \omega\} \) where the set \( H_n \) is finite and \( H_n \subset H_{n+1} \) for every \( n \in \omega \). It follows from the properties of the operator \( O \) that \( O(A) = \bigcup\{O(H_n) : n \in \omega\} \subset G(A) \).

Now assume that \( \lambda \leq \kappa \) is a cardinal and we have proved, for any \( \mu < \lambda \), that if \( A \in A \) and \( |A| \leq \mu \) then \( O(A) \subset G(A) \). Take any set \( A \in A \) with \( |A| = \lambda \); we can find an increasing \( \lambda \)-sequence \( \{B_\alpha : \alpha < \lambda\} \subset A \) such that \( A = \bigcup\{B_\alpha : \alpha < \lambda\} \) and \( |B_\alpha| < \lambda \) for any \( \alpha < \lambda \). If \( x \in O(A) \), then the property (c) of Definition 3.1
shows that there exists $\alpha < \lambda$ such that $x \in O(B_\alpha)$. By our induction hypothesis, $O(B_\alpha) \subset G(B_\alpha) \subset G(A)$ so $x \in G(A)$, i.e., we proved the inclusion $O(A) \subset G(A)$. □

3.3. Definition. Say that a family $B$ of subsets of a space $X$ is a an external pseudo-network (pseudobase) for a set $Y \subset X$ if (all elements of $B$ are open and) for any $y \in Y$ there exists a subfamily $B' \subset B$ with \{y\} = \bigcap B'. Note that an external pseudobase for $X$ is simply a $T_1$-separating family of open subsets of $X$.

3.4. Definition. Given an infinite cardinal $\kappa$, say that a space $X$ has a $\kappa$-monotone pseudo-network (pseudobase) assignment if for any set $A \subset X$ with $|A| \leq \kappa$, there exists a family $\mathcal{N}(A)$ of closed (open) subsets of $X$ such that $\mathcal{N}(A)$ is a pseudo-network (pseudobase) for the set $\overline{A}$ and the assignment $A \rightarrow \mathcal{N}(A)$ is $\kappa$-monotone. The space $X$ has a monotone pseudo-network (pseudobase) assignment if it has a $\kappa$-monotone pseudo-network (pseudobase) assignment for any infinite cardinal $\kappa$.

3.5. Proposition. Given an infinite cardinal $\kappa$, a space $X$ has a $\kappa$-monotone pseudo-network (pseudobase) assignment if and only if to any finite set $K \subset X$ we can assign a countable family $O(K)$ of closed (open) subsets of $X$ in such a way that for any set $A \subset X$ with $|A| \leq \kappa$, the family $\bigcup\{O(K) : K \in [A]^{<\omega}\}$ is a pseudo-network (pseudobase) for $\overline{A}$.

Proof. Suppose that $X$ has a $\kappa$-monotone pseudo-network (pseudobase) assignment $\mathcal{N}$. Then the family $O(K) = \mathcal{N}(K)$ is countable for any finite set $K \subset X$. Observe that $\mathcal{N}$ is defined on the family $\mathcal{A}$ of all subsets of $X$ of cardinality $\leq \kappa$ so Lemma 3.2 is applicable to see that the family $\bigcup\{O(K) : K \in [A]^{<\omega}\}$ is a pseudo-network (pseudobase) for the set $\overline{A}$.

Now assume that for any finite set $K \subset X$ we have a family $O(K)$ of closed (open) subsets of the space $X$ such that for every $A \subset X$ with $|A| \leq \kappa$, the family $\mathcal{N}(A) = \bigcup\{O(K) : K \in [A]^{<\omega}\}$ is a pseudo-network (pseudobase) for $\overline{A}$. We omit a trivial proof of the fact that the operator $\mathcal{N}$ is $\kappa$-monotone and hence it witnesses that $X$ has a $\kappa$-monotone pseudo-network (pseudobase) assignment. □

3.6. Agreement. If $\kappa$ is an infinite cardinal, $X$ is a space and $O$ is an operator defined on $[X]^{<\omega}$ with values in $[\tau(X)]^{\leq \omega}$ or $[\mathcal{L}(X)]^{<\omega}$ as in Proposition 3.5, then we will abuse notation and also call it a $\kappa$-monotone pseudobase (pseudo-network) assignment.

3.7. Observation. A space $X$ is monotonically $\kappa$-monolithic if and only if it has a $\kappa$-monotone network assignment in the sense of Definition 3.4. We will see that the concept of $\kappa$-monotone pseudo-network assignment also gives a lot of nice applications.

3.8. Proposition. If a space $X$ has a point-countable pseudobase, then $X$ has a monotone pseudobase assignment.

Proof. Fix a point-countable pseudobase $\mathcal{B}$ in $X$. If $K \subset X$ is finite, then let $O(K) = \{B \in \mathcal{B} : B \cap K \neq \emptyset\}$; it is clear that the family $O(K)$ is countable. To see that the operator $O$ is as required, take any set $A \subset X$ and a point $x \in \overline{A}$. If $y \notin X \setminus \{x\}$, then there exists $B \in \mathcal{B}$ such that $x \in B$ and $y \notin B$. It follows from $x \in \overline{A}$ that $B \cap A \neq \emptyset$. Take any $z \in B \cap A$ and observe that $B \in O(\{z\})$ and $\{z\} \in [A]^{<\omega}$. Thus, the family $\bigcup\{O(K) : K \in [A]^{<\omega}\}$ is an external pseudobase for $\overline{A}$. □

3.9. Proposition. Assume that $\kappa$ is an infinite cardinal and $X$ has a $\kappa$-monotone pseudobase assignment. Then

(a) $\psi(X) \leq \omega$;
(b) every $Y \subset X$ has a $\kappa$-monotone pseudobase assignment;
(c) if \( Y \subset X \) is separable, then \( Y \) has a countable pseudobase;
(d) if \( g : Z \to X \) is a condensation, then \( Z \) has a \( \kappa \)-monotone pseudobase assignment.

**Proof.** Let \( \mathcal{O} : [X]^{<\omega} \to [\mathcal{L}(X)]^{\leq\omega} \) be an operator of a \( \kappa \)-monotone pseudobase assignment. If \( x \in X \) then the family \( \mathcal{O}\{x\} \) is countable and contains a pseudobase at the point \( x \). Therefore \( \psi(x, X) \leq \omega \) for any \( x \in X \), i.e., we proved (a).

To see that (b) holds, it suffices to consider the family \( \mathcal{O}'(K) = \{ F \cap Y : F \in \mathcal{O}(K) \} \) for any finite \( K \subset Y \); it is immediate that \( \mathcal{O}' \) is a \( \kappa \)-monotone pseudobase assignment in \( Y \).

(c) Take a countable set \( A \subset Y \) such that \( Y \subset \overline{A} \). The family \( \mathcal{B} = \mathcal{O}(A) \) is a countable external pseudobase for \( Y \) which implies that the family \( \{ B \cap Y : B \in \mathcal{B} \} \) is a countable pseudobase in \( Y \).

To prove (d), let \( \mathcal{O}_Z(K) = \{ g^{-1}(F) : F \in \mathcal{O}(g(K)) \} \) for any finite set \( K \subset Z \). It is trivial to verify that \( \mathcal{O}_Z \) is a \( \kappa \)-monotone pseudobase assignment in \( Z \). \( \square \)

### 3.10. Proposition
Assume that \( \kappa \) is an infinite cardinal and a space \( X \) has a \( \kappa \)-monotone pseudo-network assignment. Then

(a) every \( Y \subset X \) has a \( \kappa \)-monotone pseudo-network assignment;
(b) if \( f : X \to X' \) is a perfect map, then \( X' \) has a \( \kappa \)-monotone pseudo-network assignment;
(c) if \( g : Z \to X \) is a condensation, then \( Z \) has a \( \kappa \)-monotone pseudo-network assignment.

**Proof.** If \( \mathcal{O} : [X]^{<\omega} \to [\mathcal{L}(X)]^{\leq\omega} \) is an operator of a \( \kappa \)-monotone pseudo-network assignment, then letting \( \mathcal{O}'(K) = \{ F \cap Y : F \in \mathcal{O}(K) \} \) for any finite \( K \subset Y \), we obtain an operator of a \( \kappa \)-monotone pseudo-network assignment in \( Y \); this proves (a).

To show that (b) holds, observe that we can assume, without loss of generality, that the family \( \mathcal{O}(K) \) is closed under finite intersections for any finite \( K \subset X \). For any \( z \in X' \) pick a point \( z_x \in f^{-1}(z) \) and let \( E_A = \{ z_x : z \in A \} \) for any set \( A \subset X' \). The family \( \mathcal{O}_1(K) = \{ f(P) : P \in \mathcal{O}(E_K) \} \) is countable and consists of closed subsets of \( X' \) for any finite set \( K \subset X' \). To see that \( \mathcal{O}_1 \) is an operator of a \( \kappa \)-monotone pseudo-network assignment in \( X' \) take any set \( A \subset X' \) with \( |A| \leq \kappa \). Then the set \( B = E_A \) also has cardinality \( \leq \kappa \) so the family \( \mathcal{F} = \bigcup \{ \mathcal{O}(M) : M \in [B]^{<\omega} \} \) is an external pseudo-network for \( \overline{B} \).

Observe that \( \bigcup \{ \mathcal{O}_1(K) : K \in [A]^{<\omega} \} = \{ f(P) : P \in \mathcal{F} \} \) and take an arbitrary point \( z \in \overline{A} \). The map \( f \) being closed, we have \( f(\overline{B}) = \overline{A} \) so we can take a point \( x \in \overline{B} \) such that \( f(x) = z \). Given any point \( z' \in X' \setminus \{ z \} \), the set \( Q = f^{-1}(z') \) is compact and \( x \notin Q \). Let \( \mathcal{F}_x = \{ P \in \mathcal{F} : x \in P \} \); it follows from \( \bigcap \mathcal{F}_x = \{ x \} \) and compactness of \( Q \) that there exists a finite family \( \mathcal{F}' \subset \mathcal{F}_x \) such that \( (\bigcap \mathcal{F}') \cap Q = \emptyset \). Now, \( F = \bigcap \mathcal{F}' \in \mathcal{F} \) and \( z \in f(F) \subset X' \setminus \{ z' \} \). Since \( f(F) \in \mathcal{G} = \bigcup \{ \mathcal{O}_1(K) : K \in [A]^{<\omega} \} \), we proved that the family \( \mathcal{G} \) is an external pseudo-network for \( \overline{A} \).

To finally prove (c), let \( \mathcal{O}_Z(K) = \{ g^{-1}(F) : F \in \mathcal{O}(g(K)) \} \) for any finite set \( K \subset Z \). It is standard to verify that \( \mathcal{O}_Z \) is an operator of a \( \kappa \)-monotone pseudo-network assignment in \( Z \). \( \square \)

### 3.11. Proposition
If \( X \) is a strongly monotonically \( \omega \)-monolithic space, then \( X \) has a has a monotone pseudobase assignment.

**Proof.** By Theorem 2.12 of [14], the space \( X \) is strongly monotonically monolithic which evidently, implies having a monotone pseudobase assignment. \( \square \)

### 3.12. Proposition
If \( X \) is a monotonically \( \kappa \)-monolithic space, then \( X \) has a \( \kappa \)-monotone pseudo-network assignment.

**Proof.** One of the equivalent definitions of monotone \( \kappa \)-monolithicity implies existence of an operator \( \mathcal{O} \) which assigns a countable family \( \mathcal{O}(K) \) to any finite subset \( K \subset X \) in such a way that, for any \( A \subset X \) with
$|A| \leq \kappa$, the family $\bigcup\{O(K) : K \in [A]^{<\omega}\}$ is an external network at every point of $A$. Since the space $X$ is regular, the operator $O'$ defined by $O'(K) = \{F : F \in O(K)\}$ for each finite $K \subset X$, is easily seen to witness monotone $\kappa$-monolithy so $O'$ is a $\kappa$-monotone pseudo-network assignment in $X$. \qed

3.13. Example. If $X = \beta\omega \setminus \omega$, then the space $C_p(X)$ is monotonically monolithic (see [13, Proposition 2.9]) and hence it has a monotone pseudo-network assignment by Proposition 3.12. However, Proposition 3.9 implies that $C_p(X)$ has no monotone pseudobase assignment because $\psi(C_p(X)) = d(X) > \omega$. Of course, here instead of $\beta\omega \setminus \omega$ any non-separable Lindelöf $\Sigma$-space $X$ can be taken.

Recall that a space $X$ is called perfect if every $F \in \mathcal{L}(X)$ is a $G_\delta$-set or, equivalently, every $U \in \tau(X)$ is an $F_\sigma$-set.

3.14. Proposition. If $X$ is a perfect space, then for any infinite cardinal $\kappa$, the space $X$ has a $\kappa$-monotone pseudo-network assignment if and only if it has a $\kappa$-monotone pseudobase assignment.

Proof. Suppose that $X$ has a $\kappa$-monotone pseudo-network assignment and fix, for any closed set $F \subset X$, a countable family $U_F \subset \tau(X)$ such that $\bigcap U_F = F$. Take a $\kappa$-monotone pseudo-network assignment $O : [X]^{<\omega} \to [\mathcal{L}(X)]^{<\omega}$ and let $\mathcal{G}(K) = \bigcup\{U_F : F \in O(K)\}$ for any finite subset $K \subset X$. It is clear that $\mathcal{G}$ assigns a countable family of open sets to any finite subset of $X$.

To see that $\mathcal{G}$ is a $\kappa$-monotone pseudobase assignment in $X$ take a set $A \subset X$ with $|A| \leq \kappa$ and a point $x \in A$. If $y \neq x$, then there exists a finite set $K \subset A$ and $F \in O(K)$ such that $x \in F$ and $y \notin F$. Since $F = \bigcap U_F$, there exists $U \in U_F$ such that $x \in U$ and $y \notin U$. This shows that $\{x\} = \bigcap\{U : x \in U \text{ and } U \in \mathcal{G}(K) : K \in [A]^{<\omega}\}$, i.e., $\bigcup\{\mathcal{G}(K) : K \in [A]^{<\omega}\}$ is an external pseudobase for $A$.

Now assume that $X$ has a $\kappa$-monotone pseudobase assignment and fix, for any open set $U \subset X$, a countable family $F_U$ of closed subsets of $X$ such that $\bigcup F_U = U$. Take a $\kappa$-monotone pseudobase assignment $O : [X]^{<\omega} \to [\tau(X)]^{<\omega}$ and let $\mathcal{G}(K) = \bigcup\{F_U : F \in O(K)\}$ for any finite subset $K \subset X$. It is clear that $\mathcal{G}$ assigns a countable family of closed sets to any finite subset of $X$.

To see that $\mathcal{G}$ is a $\kappa$-monotone pseudo-network assignment in $X$ take a set $A \subset X$ with $|A| \leq \kappa$ and a point $x \in A$. If $y \neq x$, then there exists a finite set $K \subset A$ and $U \in O(K)$ such that $x \in U$ and $y \notin U$. Since $U = \bigcup F_U$, there exists $F \in F_U$ such that $x \in F$; then automatically, $y \notin F$. This shows that $\{x\} = \bigcap\{F : x \in F \text{ and } F \in \bigcup\{\mathcal{G}(K) : K \in [A]^{<\omega}\}\}$, i.e., $\bigcup\{\mathcal{G}(K) : K \in [A]^{<\omega}\}$ is an external pseudo-network for $A$. \qed

3.15. Corollary. If $X$ is a perfect monotonically $\kappa$-monolithic space, then $X$ has a $\kappa$-monotone pseudobase assignment.

It is easy to see that every monotonically $\kappa$-monolithic space is $\kappa$-monolithic. However, existence of a monotone pseudobase or pseudo-network assignment in a space $X$ need not imply $\omega$-monolithy of $X$.

3.16. Example. If $X$ is the Niemytzki plane (see [4, Example 1.2.4]), then $X$ condenses onto a second countable space so it has a monotone pseudobase assignment and a monotone pseudo-network assignment by Proposition 3.8, Proposition 3.9 and Proposition 3.10 together with Proposition 3.14. However, $X$ is a separable space with $nw(X) > \omega$ and hence it is not $\omega$-monolithic.

It was proved in the paper [13] that every monotonically monolithic space has the $D$-property. The following example shows that this result cannot be extended to the spaces with a monotone pseudobase or pseudo-network assignment.
3.17. Example. In the paper [5] van Douwen and Wicke constructed an example of a space \( \Gamma \) which condenses onto \( \mathbb{R} \) without being a \( D \)-space. Applying Proposition 3.8, Proposition 3.9 and Proposition 3.10 together with Proposition 3.14 we conclude that \( \Gamma \) has both a monotone pseudobase assignment and a monotone pseudo-network assignment.

It was proved in [13] that every strongly monotonically \( \omega \)-monolithic countably compact space is compact and metrizable. The following theorem generalizes this result.

3.18. Theorem. If a countably compact space \( X \) has an \( \omega \)-monotone pseudobase assignment, then \( X \) is compact and second countable.

**Proof.** Take an \( \omega \)-monotone pseudobase assignment \( \mathcal{O} : [X]^{<\omega} \to [\tau(X)]^{<\omega} \) in the space \( X \). Pick a point \( x_0 \in X \) and let \( A_0 = \{ x_0 \} \). Proceeding inductively, assume that we have constructed increasing countable sets \( A_0, \ldots, A_n \) with the following property:

\[(1) \text{ for every number } i < n, \text{ if there exists a finite family } \mathcal{V} \subset \mathcal{O}(A_i) \text{ such that } X \setminus (\bigcup \mathcal{V}) \neq \emptyset \text{ then } (X \setminus (\bigcup \mathcal{V})) \cap A_{i+1} \neq \emptyset.\]

For every finite family \( \mathcal{V} \subset \mathcal{O}(A_n) \), if \( X \setminus \bigcup \mathcal{V} \neq \emptyset \) then choose a point \( a(\mathcal{V}) \in X \setminus \bigcup \mathcal{V} \). Let \( A_{n+1} = A_n \cup \{ a(\mathcal{V}) : \mathcal{V} \text{ is a finite subfamily of } \mathcal{O}(A_n) \text{ such that } X \setminus \bigcup \mathcal{V} \neq \emptyset \} \). It is evident that (1) is now satisfied for all \( i < n + 1 \) so we can construct a sequence \( \{ A_i : i \in \omega \} \) of countable subsets of \( X \) such that the property (1) holds for all \( n \in \omega \).

We claim that the set \( \mathcal{A} = \bigcup \{ A_n : n \in \omega \} \) is dense in \( X \). Indeed, assume that \( X \setminus \overline{\mathcal{A}} \neq \emptyset \) and pick a point \( x \in X \setminus \overline{\mathcal{A}} \). The family \( \mathcal{O}(A) \) is an external pseudobase at all points of \( \overline{\mathcal{A}} \) so we can choose, for any \( y \in \overline{\mathcal{A}} \) a set \( U_y \in \mathcal{O}(A) \) such that \( y \in U_y \) and \( x \notin U_y \). The countable cover \( \{ U_y : y \in \overline{\mathcal{A}} \} \) of the countably compact space \( \overline{\mathcal{A}} \) has a finite subcover so there exists a finite family \( \mathcal{V} \subset \mathcal{O}(A) \) such that \( x \notin \bigcup \mathcal{V} \) and \( \overline{\mathcal{A}} \subset \bigcup \mathcal{V} \).

It follows from \( \mathcal{O}(A) = \bigcup \{ \mathcal{O}(A_n) : n \in \omega \} \) and fact that the family \( \{ \mathcal{O}(A_n) : n \in \omega \} \) is increasing, that we can find \( n \in \omega \) such that \( \mathcal{V} \subset \mathcal{O}(A_n) \). Observe that \( x \in X \setminus (\bigcup \mathcal{V}) \) so the point \( y = a(\mathcal{V}) \) must belong to the set \( A_{n+1} \). However, \( y \in X \setminus (\bigcup \mathcal{V}) \subset X \setminus \overline{\mathcal{A}} \) so \( y \notin \overline{\mathcal{A}} \) which is a contradiction. Therefore \( \mathcal{A} \) is a dense subset of \( X \) so \( \mathcal{O}(A) \) is a countable pseudobase of the space. Finally apply [6, Theorem 7.6] to conclude that the space \( X \) is metrizable and hence it is compact and second countable. \( \square \)

Gruenhage proved in [7] that any compact monotonically \( \omega \)-monolithic space is Corson compact. The following theorem generalizes this result.

3.19. Theorem. If a countably compact space \( X \) has an \( \omega \)-monotone pseudo-network assignment then \( X \) is monotonically monolithic and compact. In particular, \( X \) is Corson compact.

**Proof.** Take an \( \omega \)-monotone pseudo-network assignment \( \mathcal{O} : [X]^{<\omega} \to [\mathcal{L}(\tau)]^{<\omega} \) in the space \( X \). We can assume, without loss of generality, that the family \( \mathcal{O}(K) \) is closed under finite intersections for any finite set \( K \subset X \).

Suppose that \( A \subset X \) is a countable set. We claim that the family \( \mathcal{F} = \bigcup \{ \mathcal{O}(K) : K \in [A]^{<\omega} \} \) is an external network for \( \overline{A} \). Indeed, fix a point \( x \in \overline{A} \) and \( U \in \tau(x, X) \); consider the family \( \mathcal{F}_x = \{ F \in \mathcal{F} : x \in F \} \). We have \( \bigcap \mathcal{F}_x = \{ x \} \); since \( \mathcal{F}_x \) is countable, it follows from countable compactness of \( X \setminus U \), and \( (\bigcap \mathcal{F}_x) \cap (X \setminus U) = \emptyset \) that \( (\bigcap \mathcal{F}_x) \cap (X \setminus U) = \emptyset \), i.e., \( E = \bigcap \mathcal{F} \subset U \) for some finite \( \mathcal{F} \subset \mathcal{F}_x \). However, \( E \in \mathcal{F} \) so it follows from \( x \in E \subset U \) that \( \mathcal{F} \) is an external network at \( x \).

Therefore the operator \( \mathcal{O} \) witnesses monotone \( \omega \)-monolithicity of the space \( X \). Apply [14, Corollary 2.11] to see that \( X \) is a monotonically monolithic Corson compact space. \( \square \)
If a Lindelöf \( \Sigma \)-space \( X \) has a point-countable pseudobase, then \( X \) is cosmic (see Corollary 7.10 of [6]). It turns out that it is possible to obtain the same conclusion if we assume that \( X \) has an \( \omega \)-monotone pseudobase assignment.

3.20. Theorem. If \( X \) is Lindelöf \( \Sigma \)-space with an \( \omega \)-monotone pseudobase assignment, then \( X \) is cosmic.

**Proof.** Take an \( \omega \)-monotone pseudobase assignment \( \mathcal{O} : [X]^{\leq \omega} \to [\tau(X)]^{\leq \omega} \) in the space \( X \) and choose a countable network \( \mathcal{N} \) with respect to a compact cover \( \mathcal{C} \) of the space \( X \). Pick a point \( x_0 \in X \) and let \( A_0 = \{x_0\} \). Proceeding inductively, assume that we have constructed increasing countable sets \( A_0, \ldots, A_n \) with the following property:

\[ (2) \text{ for every number } i < n, \text{ if there exists a finite family } \mathcal{V} \subset \mathcal{O}(A_i) \text{ and } N \in \mathcal{N} \text{ such that } N \backslash (\bigcup \mathcal{V}) \neq \emptyset \text{ then } (N \backslash (\bigcup \mathcal{V})) \cap A_{i+1} \neq \emptyset. \]

For every finite family \( \mathcal{V} \subset \mathcal{O}(A_n) \), if \( N \in \mathcal{N} \) and \( N \backslash \bigcup \mathcal{V} \neq \emptyset \) then choose a point \( a(\mathcal{V}, N) \in N \backslash \bigcup \mathcal{V} \). Let \( A_{n+1} = A_n \cup \{a(\mathcal{V}, N) : N \in \mathcal{N} \text{ and } \mathcal{V} \text{ is a finite subfamily of } \mathcal{O}(A_n) \text{ such that } N \backslash \bigcup \mathcal{V} \neq \emptyset\} \). It is evident that \( (2) \) is now satisfied for all \( i < n + 1 \) so we can construct a sequence \( \{A_i : i \in \omega\} \) of countable subsets of \( X \) such that the property \( (2) \) holds for all \( n \in \omega \).

We claim that the set \( A = \bigcup\{A_n : n \in \omega\} \) is dense in \( X \). Striving for a contradiction, assume that \( X \backslash \overline{A} \neq \emptyset \) and pick a point \( x \in X \backslash \overline{A} \). There exists a set \( C \subset \mathcal{C} \) with \( x \in C \). The set \( K = C \cap \overline{A} \) is compact and the family \( \mathcal{O}(A) \) is an external pseudobase at all points of \( \overline{A} \) so we can choose a finite family \( \mathcal{V} \subset \mathcal{O}(A) \) such that \( x \notin \bigcup \mathcal{V} \) and \( K \subset \bigcup \mathcal{V} \). Observe that \( G = (\bigcup \mathcal{V}) \cup (X \backslash \overline{A}) \) is an open neighborhood of \( C \) and therefore we can find \( N \in \mathcal{N} \) such that \( C \subset N \subset G \).

It follows from \( \mathcal{O}(A) = \bigcup\{\mathcal{O}(A_n) : n \in \omega\} \) and fact that the family \( \{\mathcal{O}(A_n) : n \in \omega\} \) is increasing, that we can find \( n \in \omega \) such that \( V \subset \mathcal{O}(A_n) \). Observe that \( x \in N \backslash (\bigcup \mathcal{V}) \) so the point \( y = a(\mathcal{V}, N) \) must belong to the set \( A_{n+1} \). However, \( y \in N \backslash (\bigcup \mathcal{V}) \subset X \backslash \overline{A} \) so \( y \notin \overline{A} \) which is a contradiction. Therefore \( A \) is a dense subset of \( X \) so \( \mathcal{O}(A) \) is a countable pseudobase of the space \( X \). Finally apply Corollary 7.10 of [6] to conclude that the space \( X \) has a countable network. \( \square \)

3.21. Corollary. If a Lindelöf \( \Sigma \)-space \( X \) can be condensed onto a space with an \( \omega \)-monotone pseudobase assignment, then \( X \) is cosmic.

**Proof.** By Proposition 3.9, the space \( X \) has an \( \omega \)-monotone pseudobase assignment so \( X \) must be cosmic by Theorem 3.20. \( \square \)

3.22. Theorem. For an arbitrary infinite cardinal \( \kappa \), a Lindelöf \( \Sigma \)-space \( X \) has a \( \kappa \)-monotone pseudo-network assignment if and only if \( X \) is monotonically \( \kappa \)-monolithic.

**Proof.** Since sufficiency is immediate from Proposition 3.12, assume that \( X \) has a \( \kappa \)-monotone pseudo-network assignment \( \mathcal{O} : [X]^{< \kappa} \to [\mathcal{L}(X)]^{< \kappa} \). Observe that if, for any finite set \( K \subset X \), we add to the family \( \mathcal{O}(K) \) all elements of \( \mathcal{O}(L) \) for all sets \( L \subset K \), then we obtain a countable family and the respective modified operator will still be a \( \kappa \)-monotone pseudo-network assignment in \( X \).

Therefore, we can assume, without loss of generality, that \( L \subset K \) implies \( \mathcal{O}(L) \subset \mathcal{O}(K) \) and every family \( \mathcal{O}(K) \) is closed under finite intersections.

There exists a compact cover \( \mathcal{C} \) of the space \( X \) such that some countable family \( \mathcal{N} \) of subsets of \( X \) is a network with respect to \( \mathcal{C} \). For any finite set \( K \subset X \) let \( \mathcal{G}(K) = \{G \cap N : G \in \mathcal{O}(K) \text{ and } N \in \mathcal{N}\} \); the family \( \mathcal{G}(K) \) is clearly countable so it suffices to show that \( \mathcal{G} \) witnesses monotone \( \kappa \)-monolithicity of the space \( X \). To this end, take a set \( A \subset X \) with \( |A| \leq \kappa \), a point \( x \in \overline{A} \) and a set \( U \in \tau(x, X) \). Pick \( C \in \mathcal{C} \) with \( x \in C \); let \( \mathcal{H} = \bigcup\{\mathcal{O}(K) : K \subset [A]^{< \omega}\} \) and \( \mathcal{H}_x = \{H \in \mathcal{H} : x \in H\} \). Since \( C' = C \setminus U \) is compact, it
follows from \( \bigcap \mathcal{H}_x = \{ x \} \) that we can find a finite family \( \mathcal{F} \subset \mathcal{H}_x \) such that \( (\bigcap \mathcal{F}) \cap C' = \emptyset \). There exists a finite set \( K \subset A \) such that \( \mathcal{F} \subset \mathcal{O}(K) \).

By normality of \( X \), we can find disjoint sets \( V, W \in \tau(X) \) such that \( C' \subset V \) and \( F = \bigcap \mathcal{F} \subset W \). If \( N \in \mathcal{N} \) is a set such that \( C \subset N \subset U \cup V \), then \( N' = \bigcap \mathcal{F} \subset \mathcal{G}(K) \) and \( x \in N' \). This shows that \( \bigcup \{ \mathcal{G}(K) : K \in [A]^{< \omega} \} \) is an external network at every point of \( A \), i.e., the space \( X \) is monotonically \( \kappa \)-monolithic. \( \Box \)

3.23. Proposition. Given an infinite cardinal \( \kappa \) suppose that \( X \) is a space such that \( t(X) \leq \kappa \). Then

(a) existence of a \( \kappa \)-monotone pseudo-network assignment in \( X \) implies that \( X \) has a monotone pseudo-network assignment;
(b) existence of a \( \kappa \)-monotone pseudobase assignment in \( X \) implies that \( X \) has a monotone pseudobase assignment.

Proof. We will prove the statements of (a) and (b) simultaneously. Let \( \mathcal{P} \) denote the fact of existence of a monotone pseudobase (or a pseudo-network) assignment in \( X \) and let \( \mathcal{P}_\kappa \) be its respective \( \kappa \)-version. Fix an operator \( \mathcal{O} \) defined on \( [X]^{< \omega} \) that witnesses \( \mathcal{P}_\kappa \) and take an arbitrary set \( A \subset X \). If \( x \in \bar{A} \) and \( y \neq x \), then we can find a set \( B \subset A \) such that \( |B| \leq \kappa \) and \( x \in \overline{B} \). Since \( \mathcal{O} \) witnesses \( \mathcal{P}_\kappa \), we can find a finite set \( K \subset B \) for which \( x \in F \neq y \) for some \( F \in \mathcal{O}(K) \). Therefore the family \( \bigcup \{ \mathcal{O}(K) : K \in [A]^{< \omega} \} \) is a pseudobase (or a pseudo-network respectively) at every point of \( A \), i.e., \( \mathcal{O} \) also witnesses \( \mathcal{P} \). \( \Box \)

3.24. Theorem. For any infinite cardinal \( \kappa \),

(a) if \( X_n \) has a \( \kappa \)-monotone pseudo-network assignment for every \( n \in \omega \), then the space \( X = \prod_{n \in \omega} X_n \) has a \( \kappa \)-monotone pseudo-network assignment;
(b) if \( X_n \) has a \( \kappa \)-monotone pseudobase assignment for every \( n \in \omega \), then the space \( X = \prod_{n \in \omega} X_n \) has a \( \kappa \)-monotone pseudobase assignment.

Proof. We will prove the statements of (a) and (b) simultaneously. Let \( \mathcal{P}_\kappa \) denote the fact of existence of a \( \kappa \)-monotone pseudobase (or a pseudo-network) assignment in \( X \). Denote by \( p_n \) the projection of the space \( X \) onto \( X_n \) and take an operator \( \mathcal{O}_n \) that witnesses the property \( \mathcal{P}_\kappa \) in the space \( X_n \) for each \( n \in \omega \). Given a finite set \( K \subset X \), let \( \mathcal{O}(K) = \{ p_n^{-1}(F) : n \in \omega \text{ and } F \in \mathcal{O}_n(p_n(K)) \} \). It is clear that the family \( \mathcal{O}(K) \) is countable and consists of closed or open subsets of \( X \) respectively.

To see that \( \mathcal{O} \) witnesses the property \( \mathcal{P}_\kappa \) in \( X \) take a set \( A \subset X \) with \( |A| \leq \kappa \) and let \( A_n = p_n(A) \) for any \( n \in \omega \). If \( x \in \bar{A} \) and \( y \neq x \), then there exists \( n \in \omega \) such that \( x(n) \neq y(n) \); since \( x(n) \in \bar{A}_n \), we can find a finite set \( Q \subset A_n \) such that \( x(n) \in F \neq y(n) \) for some \( F \in \mathcal{O}_n(Q) \). Take a finite set \( K \subset A \) such that \( Q = p_n(K) \) and observe that \( G = p_n^{-1}(F) \in \mathcal{O}(K) \). Since also \( x \in G \neq y \), we proved that \( \mathcal{O} \) witnesses the property \( \mathcal{P}_\kappa \) in \( X \). \( \Box \)

3.25. Theorem. Given an infinite cardinal \( \kappa \), every \( \sigma \)-product of spaces with a \( \kappa \)-monotone pseudo-network assignment has a \( \kappa \)-monotone pseudo-network assignment.

Proof. Suppose that a space \( X_t \) has a \( \kappa \)-monotone pseudo-network assignment for every \( t \in T \) and fix a point \( a \in X = \prod \{ X_t : t \in T \} \). We must prove that the \( \sigma \)-product \( Y = \{ x \in X : |\{ t \in T : x(t) \neq a(t) \} | < \omega \} \) also has a \( \kappa \)-monotone pseudo-network assignment.

For each \( t \in T \) let \( p_t : Y \to X_t \) be the projection and choose a \( \kappa \)-monotone pseudo-network assignment \( \mathcal{O}_t : [X_t]^{< \omega} \to [\mathcal{L}(X_t)]^{< \omega} \) in the space \( X_t \). There is no loss of generality to assume that \( X_t \in \mathcal{O}_t(K) \subset \mathcal{O}_t(L) \) for any finite subsets \( K, L \) of the space \( X_t \) such that \( K \subset L \). For every \( x \in Y \) denote by \( \text{supp}(x) \) the set
\{t \in T : x(t) \neq a(t)\}. Given a set \(S \subset T\), we will need the point \(a_S \in \prod_{t \in S} X_t\) defined by \(a_S(t) = a(t)\) for any \(t \in S\).

For any finite set \(K \subset Y\) let \(S = \bigcup \{\text{supp}(x) : x \in K\}\) and consider the family \(\mathcal{H}(K, S') = \{\prod_{t \in S'} P_t \times \{a_{T \setminus S'}\} : P_t \in \mathcal{O}_t(p_t(K))\}\) for every \(t \in S'\) for any \(S' \subset S\); letting \(\mathcal{O}(K) = \{\{a\}\} \cup \bigcup \{\mathcal{H}(K, S') : S' \subset S\}\), we assign a countable family of closed subsets of \(Y\) to any finite set \(K \subset Y\).

To see that \(\mathcal{O}\) is a \(\kappa\)-monotone pseudo-network assignment take any set \(A \subset Y\), a point \(x \in \overline{A}\) and \(y \in Y \setminus \{x\}\). Since \(\{a\} \in \mathcal{O}(K)\) for any finite \(K \subset Y\), we can assume that \(x \neq a\) and hence the set \(\text{supp}(x)\) is non-empty. It is easy to see that \(\text{supp}(x) \subset S = \bigcup \{\text{supp}(z) : z \in A\}\).

There exists an index \(t \in T\) such that \(x(t) \neq y(t)\); it follows from \(x(t) \in \overline{p_t(A)}\) that there exists a finite set \(K \subset A\) such that \(\text{supp}(x) \subset K\) and \(x(t) \in F \neq y(t)\) for some set \(F \in \mathcal{O}_t(p_t(K))\).

If \(t \in S\), then let \(F_s = X_s\) for every \(s \in \text{supp}(x) \setminus \{t\}\); it is immediate that the set \(P = F \times \prod_{s \in \text{supp}(x) \setminus \{t\}} \{a_{T \setminus \text{supp}(x)}\}\) belongs to \(\mathcal{O}(K)\) so it follows from the inclusions \(x \in P \subset Y \setminus \{y\}\) that the family \(\bigcup \{\mathcal{O}(L) : L \in [A]^{<\omega}\}\) separates \(x\) from \(y\).

Now, if \(t \notin S\), then let \(F_s = X_s\) for any \(s \in \text{supp}(x)\). As before, the set \(P = \prod_{s \in \text{supp}(x)} \{a_{T \setminus \text{supp}(x)}\}\) belongs to \(\mathcal{O}(K)\) and it follows from \(x \in P \subset Y \setminus \{y\}\) that the family \(\bigcup \{\mathcal{O}(L) : L \in [A]^{<\omega}\}\) separates \(x\) from \(y\). \(\Box\)

3.26. Example. Consider the \(\Sigma\)-product \(X = \{x \in \mathbb{R}^{\omega_1} : x^{-1}(\mathbb{R}\setminus\{0\})\}\) is countable) of \(\omega_1\)-many real lines. It is easy to see that the ordinal \(\omega_1\) with the interval topology embeds in the space \(X\). Since \(\omega_1\) is a countably compact non-compact space, it does not have an \(\omega\)-monotone pseudo-network assignment by Theorem 3.19 and hence \(X\) does not have such an assignment either by Proposition 3.10. Therefore \(\Sigma\)-products of spaces with an \(\omega\)-monotone pseudo-network assignment need not have an \(\omega\)-monotone pseudo-network assignment.

3.27. Proposition. If \(X\) has a dense Lindelöf \(\Sigma\)-subspace, then \(C_p(X)\) has a monotone pseudo-network assignment.

Proof. Fix a dense \(D \subset X\) with the Lindelöf \(\Sigma\)-property; then the restriction map \(\pi : C_p(X) \to C_p(D)\) is injective and hence \(C_p(X)\) condenses onto the set \(Z = \pi(C_p(X)) \subset C_p(D)\). The space \(C_p(D)\) is monotonically monolithic by Proposition 2.9 of [13] and hence so is \(Z\). By Proposition 3.12 the space \(Z\) has a monotone pseudo-network assignment so Proposition 3.10 completes the proof. \(\Box\)

3.28. Proposition. If \(X\) is a monotonically monolithic space, then \(C_pC_p(X)\) has a monotone pseudo-network assignment.

Proof. It follows from [11, Theorem 3.7] that \(C_pC_p(X)\) is also monotonically monolithic so Proposition 3.12 completes the proof. \(\Box\)

3.29. Corollary. If \(X\) is a space with a point-countable base, then \(C_pC_p(X)\) has a monotone pseudo-network assignment.

Proof. Just observe that any space with a point-countable base is monotonically monolithic (see Proposition 2.5 of [13]) and apply Proposition 3.28. \(\Box\)

The following theorem answers Question 4.2 from [12].

3.30. Theorem. Suppose that \(X\) is a space such that \(s(X) \leq \omega\) and \(C_pC_p(X)\) has the Lindelöf \(\Sigma\)-property. Then \(X\) is cosmic.

Proof. The Lindelöf \(\Sigma\)-property of \(C_pC_p(X)\) implies that \(C_pC_p(X)\) is monotonically \(\omega\)-monolithic (see [1, Corollary 2.16]) and hence \(\omega\)-monolithic. Applying Proposition 4 of [3] we conclude that \(X\) is hereditarily
Lindelöf and hence perfect. Corollary 3.15 shows that \( X \) has an \( \omega \)-monotone pseudobase assignment. The space \( X \) must have the Lindelöf \( \Sigma \)-property being homeomorphic to a closed subset of \( C_pC_p(X) \), so we can apply Theorem 3.20 to see that \( X \) is cosmic. \( \square \)

4. Open problems

The class of spaces with a monotone pseudo-network assignment appears for the first time in this work, but the author hopes that the obtained results show that it is a nice and useful class. To show the reader that what was done in this paper has just scratched the surface of a big topic, we list some interesting open questions below.

4.1. Question. Does there exist a space \( X \) with an \( \omega \)-monotone pseudobase assignment in which there is no \( \omega \)-monotone pseudo-network assignment?

4.2. Question. Suppose that \( X \) is a linearly ordered topological space with an \( \omega \)-monotone pseudobase assignment. Must \( X \) have an \( \omega \)-monotone pseudo-network assignment?

4.3. Question. Suppose that \( X \) is a linearly ordered topological space with an \( \omega \)-monotone pseudobase assignment. Must \( X \) have a point-countable pseudobase?

4.4. Question. Suppose that \( X \) is a linearly ordered topological space with an \( \omega \)-monotone pseudobase assignment. Must \( X \) be a D-space?

4.5. Question. Suppose that \( X \) is a pseudocompact space with an \( \omega \)-monotone pseudobase assignment. Must \( X \) have a point-countable pseudobase?

4.6. Question. Suppose that \( X \) has an \( \omega \)-monotone pseudobase assignment and \( \omega_1 \) is a caliber of \( X \). Must \( X \) have a countable pseudobase?

4.7. Question. Suppose that \( X \) is a subspace of the ordinal \( \omega_1 \) with the interval topology and \( X \) has an \( \omega \)-monotone pseudobase assignment. Must \( X \) be metrizable?

4.8. Question. Suppose that \( X \) has an \( \omega \)-monotone pseudo-network assignment and \( f: X \to Y \) is a continuous closed onto map. Must \( Y \) have an \( \omega \)-monotone pseudo-network assignment?

4.9. Question. Suppose that \( X \) has an \( \omega \)-monotone pseudo-network assignment. Must \( C_pC_p(X) \) have an \( \omega \)-monotone pseudo-network assignment?

4.10. Question. Suppose that \( X \) has a countable pseudobase. Must \( C_pC_p(X) \) have an \( \omega \)-monotone pseudo-network assignment?

4.11. Question. Suppose that \( X \) has a point-countable pseudobase. Must \( C_pC_p(X) \) have an \( \omega \)-monotone pseudo-network assignment?

References