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DISCRETE REFLEXIVITY IN SQUARES

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Abstract. We establish that first countability is discretely reflexive in the class of pseudocompact spaces of character at most ω_1. Given a property $\mathcal{P}$ from the list {σ-compactness, zero-dimensionality, analyticity}, it is shown that a space X with a countable network has $\mathcal{P}$ if and only if $\overline{D}$ has $\mathcal{P}$ for any discrete set $D \subset X \times X$. If X is a Lindelöf Σ-space and $\overline{D}$ is countable for every discrete subspace $D \subset X \times X$, then X is countable. However, under CH, there exists an example of an uncountable space X such that $\overline{D}$ is countable for any discrete $D \subset X \times X$. We also prove that for any space X of countable π-weight, there exists a discrete $D \subset X \times X$ such that $\Delta_X = \{(x, x) : x \in X\} \subset \overline{D}$. Therefore, if X has countable π-weight, then for every closed-hereditary property $\mathcal{P}$, if $X \times X$ is discretely $\mathcal{P}$, then the space X has $\mathcal{P}$.

1. Introduction

To find out whether a topological space X has a given property $\mathcal{P}$ it is often useful to analyze small subspaces of X. For example, a countably compact space X is metrizable if and only if all subspaces of X of cardinality at most ω_1 are metrizable; this is a result of A. Dow (see [11]). Therefore, to check whether X is metrizable it suffices to verify metrizability of "small" subspaces of X, i.e., to prove that every subspace $Y \subset X$ with $|Y| \leq \omega_1$ is metrizable. Hajnal and Juhász established in [15], among other things, that countable weight also reflects

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in small subspaces, i.e., a space X is second countable if and only if each subspace of X of cardinality $\leq \omega_1$ is second countable.

For most spaces X, the closures of discrete subsets of X can also be considered small and it is not always evident whether they represent the properties of the whole space. Recall that a property $\mathcal{P}$ is called *discretely reflexive* if a space X has $\mathcal{P}$ whenever it is *discretely* $\mathcal{P}$, i.e., $\overline{D}$ has $\mathcal{P}$ for any discrete $D \subset X$.

Tkachuk established in [18] that a space X is compact if and only if the closure of every discrete subspace of X is compact, i.e., compactness is a discretely reflexive property. Alas, Tkachuk and Wilson proved in [1] that quite a few cardinal functions are reflected by closures of discrete sets in compact spaces. For example, a compact space X has countable character (tightness) if and only if the closure of every discrete subspace of X has countable character (or countable tightness respectively).

The paper [7] provided some results on discrete reflexivity for countably compact spaces generalizing the respective theorems of [1] established for compact spaces. Such a generalization turned out to possible for the spaces of character $\leq \omega_1$. It was proved, in particular, that countable tightness and countable character are discretely reflexive in countably compact spaces of weight $\leq \omega_1$. It was also shown in [7] that, at least consistently, countable compactness cannot be omitted in these results.

It is an open problem of Arhangel'skii [5, Problem 14] whether the Lindelöf property is discretely reflexive. Arhangel'skii and Buzyakova established in [6] that any discretely Lindelöf space of countable tightness is Lindelöf. Tkachuk and Wilson proved in [19] that paracompactness and Lindelöfness are both discretely reflexive in GO spaces.

If we consider completeness or convergence properties then one cannot expect positive results on discrete reflexivity in general spaces. Indeed, van Douwen's example of countable maximal space [10, Example 3.3] shows that a space X need not be sequential or have the Baire property even if every discrete subspace in X is closed and hence X is discretely metrizable and discretely Čech-complete at the same time.

It turns out that some non-discretely reflexive properties improve their behavior in $X \times X$ for an arbitrary X. It was proved in [7] that for any countably compact space X, if $\overline{D}$ is metrizable for any discrete $D \subset X \times X$ then X is metrizable and hence compact. A result from [8] states that, for any topological property $\mathcal{P}$, preserved by continuous maps, if X is compact and $\overline{D}$ has $\mathcal{P}$ for any discrete $D \subset X^2$ then X has $\mathcal{P}$. It is also established in [8] that for every Lindelöf p-space X there exists a discrete $D \subset X \times X$ such that $\Delta_X \subset \overline{D}$ and therefore the projection of the set D onto the first coordinate is dense in X; in particular, Δ_X is a retract of $\overline{D}$. Here $\Delta_X = \{(x,x) : x \in X\}$ is the diagonal of the space X. Another example constructed under CH in [8], shows that a countable space X can be maximal while all discrete subspaces of $X \times X$ are closed. Therefore, under CH, the diagonal of a countable space is not necessarily contained in the closure of a discrete set.

In this paper we generalize Theorem 2.3 of [7] showing that for every regular feebly compact space of character ω_1, if $\overline{D}$ is first countable for any discrete $D \subset X$ then X is first countable. Under CH, we show that there exists a non-σ-compact space X such that $\overline{D}$ is countable for any discrete $D \subset X \times X$; this consistently answers Problem 3.4 and Problem 3.6 of [8].

We show that, for any space X with a countable network, there exists a discrete set $D \subset X \times X$ such that $\Delta \backslash \overline{D}$ is countable. As an easy consequence, if X has a countable network and the set $\overline{D}$ is σ-compact (analytic) for any discrete $D \subset X \times X$ then X is σ-compact (analytic respectively). This answers Problems 3.8 and 3.10 from [8].

We also prove that for any space X of countable π-weight, there exists a discrete set $D \subset X \times X$ such that $\Delta_X \subset \overline{D}$. This implies that quite a few topological properties are discretely reflexive in the squares of spaces with a countable π-base.

2. Notation and terminology.

All spaces are assumed to be Hausdorff. Given a space X, the family $\tau(X)$ is its topology and $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any $x \in X$; if $A \subset X$ then $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. All ordinals are identified with the set of their predecessors and are assumed to carry the interval topology. We denote by $\mathbb{D}$ the set $\{0, 1\}$ with the discrete topology. The set $\mathbb{R}$ is the real line with its usual topology, $\mathbb{N} = \omega \backslash \{0\}$ and $\mathbb{Q} \subset \mathbb{R}$ is the set of rationals. If X is a space and $Y \subset X$ then $\Delta_Y = \{(x,x) : x \in Y\} \subset X \times X$ is the diagonal of Y considered as a subspace of $X \times X$.

A space X is *feebly compact* if every locally finite family of non-empty open subsets of X is finite. The space X is *pseudocompact* if it is feebly compact and Tychonoff. Given an infinite cardinal κ, call a space X *initially κ-compact* if every open cover of X of cardinality $\leq \kappa$ has a finite subcover. If $\mathcal{P}$ is a topological property then X is *discretely* $\mathcal{P}$ if $\overline{D}$ has $\mathcal{P}$ for any discrete $D \subset X$.

Say that a family $\mathcal{F}$ of subsets of a space X is *a network modulo a cover* $\mathcal{C}$ if for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$ there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$. A space X is *Lindelöf* Σ (or has *the Lindelöf* Σ*-property*) if there exists a countable family $\mathcal{F}$ of subsets of X such that $\mathcal{F}$ is a network modulo a compact cover $\mathcal{C}$ of the space X. Say that X is a *Lindelöf p-space* if X can be perfectly mapped onto a second countable space.

As usual, we denote by $d(X)$ the minimal cardinality of a dense subset of X and $hd(X) = \sup\{d(Y) : Y \subset X\}$. The minimal cardinality of a local base at a point $x \in X$ is called the *character of X at x*; it is denoted by $\chi(x, X)$ and $\chi(X) = \sup\{\chi(x, X) : x \in X\}$. If X is a space and $x \in X$ then let $\psi(x, X) = \min\{|\mathcal{U}| : \mathcal{U} \subset \tau(X) \text{ and } \bigcap \mathcal{U} = \{x\}\}$ and $\psi(X) = \sup\{\psi(x, X) : x \in X\}$; the cardinal $\psi(X)$ is called the *pseudocharacter* of the space X. Given an infinite cardinal κ we say that $t(X) \leq \kappa$ if, for any $A \subset X$ and $x \in \overline{A}$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$. Now, $hl(X) = \sup\{l(Y) : Y \subset X\}$ is the hereditary Lindelöf number of X. The cardinal $iw(X) = \min\{\kappa :$ the space X has a weaker topology of weight $\kappa\}$ is called the *i-weight* of X.

We say that X is *a strong L-space* if $d(X) > \omega$ but $hl(X^n) \leq \omega$ for all $n \in \mathbb{N}$ Given a space X, a family $\mathcal{N}$ of subsets of X is a network of X if for every $U \in \tau(X)$ there exists a family $\mathcal{N}' \subset \mathcal{N}$ such that $U = \bigcup \mathcal{N}'$. Furthermore, $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a network in $X\}$. The cardinal $nw(X)$ is called the *network weight* of X.

The rest of our terminology is standard and follows [13]; the survey of Hodel [16] can be consulted for definitions and properties of cardinal invariants.

3. Discrete reflexivity and feeble compactness

The purpose of this section is to show that some convergence properties are discretely reflexive in feebly compact regular spaces; this will generalize the respective results from [7] about countably compact spaces.

The next two statements are part of the folklore; their proofs are straightforward and can be left to the reader.

Lemma 3.1. *If X is a regular feebly compact space and $\psi(x, X) \leq \omega$ for some $x \in X$ then $\chi(x, X) \leq \omega$.*

Lemma 3.2. *Given an infinite cardinal κ, if X is a regular initially κ-compact space and $\psi(x, X) \leq \kappa$ for some $x \in X$ then $\chi(x, X) \leq \kappa$.*

The following result strengthens Theorem 2.3 of [7]; it is also new for pseudocompact spaces. We formulate it together with Theorem 3.4 because their proofs are very similar and can be given simultaneously.

Theorem 3.3. *If X is a regular feebly compact space of character $\leq \omega_1$ then the following conditions are equivalent:*

(a) $\overline{D}$ *has countable pseudocharacter for any discrete* $D \subset X$*;*
(b) $\overline{D}$ *is first countable for any discrete* $D \subset X$*;*
(c) X *is first countable.*

Theorem 3.4. *Given an infinite cardinal* κ*, if* X *is a regular initially* κ*-compact space of character* $\leq \kappa^+$ *then the following conditions are equivalent:*
(a) $\psi(\overline{D}) \leq \kappa$ *for any discrete* $D \subset X$*;*
(b) $\chi(\overline{D}) \leq \kappa$ *for any discrete* $D \subset X$*;*
(c) $\chi(X) \leq \kappa$*.*

Proof of Theorems 3.3 and 3.4. In both proofs it suffices to establish the implication (a)$\Longrightarrow$(c). The proof given below works for Theorem 3.4; if we replace κ by ω then we obtain the proof of Theorem 3.3.

Assume that $\psi(\overline{D}) \leq \kappa$ for any discrete set $D \subset X$ and fix an arbitrary point $x \in X$. If $\psi(x, X) \leq \kappa$ then $\chi(x, X) \leq \kappa$ by Lemma 3.2 (or Lemma 3.1 respectively) so we can assume that $\psi(x, X) = \kappa^+$.

Take a local base $\{V_\alpha : \alpha < \kappa^+\}$ at the point x. For any $\beta < \kappa^+$ consider the set $F_\beta = \bigcap\{\overline{V}_\alpha : \alpha < \beta\}$. It follows from $\psi(x, X) > \kappa$ that $\{F_\beta \backslash \{x\} : \beta < \kappa^+\}$ is a decreasing family of non-empty closed subsets of the space $X \backslash \{x\}$.

Apply Lemma 2.3 of the paper [1] to find a discrete set $D \subset X \backslash \{x\}$ such that $\overline{D} \cap (F_\beta \backslash \{x\}) \neq \emptyset$ for every $\beta < \kappa^+$. It is immediate that $x \in \overline{D}$; so, we can find a κ-sequence $\{U_\alpha : \alpha < \kappa\}$ of open subsets of X such that $(\bigcap_{\alpha<\kappa} U_\alpha) \cap \overline{D} = \{x\}$. For every $\alpha < \kappa$ apply regularity of the space X to find an ordinal $\gamma_\alpha < \kappa^+$ such that $\overline{V}_{\gamma_\alpha} \subset U_\alpha$. If $\gamma = \sup\{\gamma_\alpha : \alpha < \kappa\}$ then $F_\gamma \subset \bigcap_{\alpha<\kappa} U_\alpha$ and therefore $F_\gamma \cap \overline{D} = \{x\}$; this contradiction with $(F_\gamma \backslash \{x\}) \cap \overline{D} \neq \emptyset$ shows that $\chi(x, X) \leq \kappa$ for any $x \in X$. $\square$

Example 3.5. Under CH it is possible to find a dense Luzin subspace L of the Σ-product $S = \{x \in \mathbb{D}^{\omega_1} : |x^{-1}(1)| \leq \omega\}$; consider the point $u \in \mathbb{D}^{\omega_1}$ defined by $u(\alpha) = 1$ for any $\alpha < \omega_1$. The space L is easily seen to be hereditarily Lindelöf and hence so is $X = L \cup \{u\}$. If $D \subset L$ is a discrete subset then $|D| \leq \omega$ and it follows from $D \subset S$ that $\mathrm{cl}_{\mathbb{D}^{\omega_1}}(D) \subset S$, i.e., the set $\mathrm{cl}_X(D)$ is separable and metrizable for any discrete $D \subset X$. However, $w(X) = \omega_1$ while $t(X) > \omega$ and hence $\chi(X) > \omega$; this shows that we cannot replace feeble compactness with hereditary Lindelöf property in Theorem 3.3.

Example 3.6. Let τ be the usual metric topology on the interval $[0, 1] \subset \mathbb{R}$ and consider an ultrafilter $\mathcal{F}$ of dense sets of $([0, 1], \tau)$ which contains $\mathbb{Q}$. Denote by μ the topology on $[0, 1]$ generated by $\tau \cup \mathcal{F}$; it is easy to see that $X = ([0, 1], \mu)$ is a submaximal space and hence every discrete $D \subset X$ is closed in X. This implies that X is discretely metrizable and therefore $\mathrm{cl}_X(D) = D$ has countable character for any discrete $D \subset X$. However, X has no non-trivial convergent sequences and hence $\chi(X) > \omega$. Since $\mathcal{F}$ contains a filterbase of size $\mathfrak{c}$, we have $w(X) \leq \mathfrak{c}$. It is standard to show that the space $X = ([0,1], \mu)$ is H-closed and hence feebly compact so under CH there exists a feebly compact Hausdorff non-regular space X of weight ω_1 which is discretely first countable but not first countable.

The following result generalizes Theorem 2.3 of the paper [7].

Theorem 3.7. *Given a cardinal $\kappa \geq \omega$, suppose that X is a regular initially κ-compact space with $\chi(X) \leq \kappa^+$. If $t(\overline{D}) \leq \kappa$ for every discrete set $D \subset X$ then $t(X) \leq \kappa$.*

Proof. If $t(X) > \kappa$ then we can find a non-closed set $A \subset X$ such that $\overline{B} \subset A$ for any $B \subset A$ with $|B| \leq \kappa$. It is straightforward that the set A must be initially κ-compact; fix a point $x \in \overline{A} \backslash A$.

If $\psi(x, A \cup \{x\}) \leq \kappa$ then there exists a family $\{U_\alpha : \alpha < \kappa\} \subset \tau(x, X)$ such that $\bigcap\{\overline{U}_\alpha : \alpha < \kappa\} \cap A = \emptyset$. Therefore $\mathcal{F} = \{\overline{U}_\alpha \cap A : \alpha < \kappa\}$ is a centered family of non-empty closed subsets of A with empty intersection. Since $|\mathcal{F}| \leq \kappa$, we have a contradiction with initial κ-compactness of A; this shows that $\psi(x, A \cup \{x\}) = \chi(x, A \cup \{x\}) = \kappa^+$ so we can apply Lemma 2.2 of [2] to conclude that there is a discrete set $D \subset A$ such that $x \in \overline{D}$. Now it follows from $t(\overline{D}) \leq \kappa$ that there is a set $B \subset D$ such that $|B| \leq \kappa$ and $x \in \overline{B}$; this final contradiction proves that $t(X) \leq \kappa$. □

The next proposition should be compared with Lemma 2.2 of [2] and may be useful for the study of discrete reflexivity of pseudocharacter.

Proposition 3.8. *Given a regular cardinal κ, assume that X is a regular space and $\psi(p, X) = \chi(p, X) = \kappa$ for some $p \in X$. Then there is a discrete set $D \subset X \backslash \{p\}$ such that $p \in \overline{D}$ and $\psi(p, \overline{D}) = \kappa$.*

Proof. Let $\{V_\alpha : \alpha < \kappa\}$ be a local base at the point p. If $F_\alpha = \bigcap\{\overline{V}_\beta : \beta < \alpha\}$ for each $\alpha < \kappa$ then $\{F_\alpha \backslash \{p\} : \alpha < \kappa\}$ is a decreasing family of non-empty closed subsets of $X \backslash \{p\}$. Apply Lemma 2.3 of [1] to find a discrete set $D \subset X \backslash \{p\}$ such that $\overline{D} \cap (F_\alpha \backslash \{p\}) \neq \emptyset$ for any $\alpha < \kappa$ and hence $p \in \overline{D}$.

If $\psi(p, \overline{D}) = \lambda < \kappa$ then we can choose a family $\{U_\alpha : \alpha < \lambda\} \subset \tau(p, X)$ such that $(\bigcap_{\alpha<\lambda} U_\alpha) \cap \overline{D} = \{p\}$. For every $\alpha < \lambda$ pick an ordinal $\gamma_\alpha < \kappa$ such that $\overline{V}_{\gamma_\alpha} \subset U_\alpha$. The cardinal κ being regular, there exists an ordinal $\mu < \kappa$ such that $\sup\{\gamma_\alpha : \alpha < \lambda\} < \mu$. As a consequence, $F_\mu \subset \bigcap_{\alpha<\lambda} U_\alpha$ so $F_\mu \cap \overline{D} = \{p\}$ which contradicts $\overline{D} \cap (F_\mu \backslash \{p\}) \neq \emptyset$; therefore $\psi(p, \overline{D}) = \kappa$. □

Given a space X call a discrete set $D \subset X \times X$ *large* if $|D| = d(X)$. In [17] Juhász and Szentmiklóssy established that if X is a space and there exists a large discrete $D \subset X \times X$ then X^ω is d-separable. They also proved that if X is compact then a large discrete set can be found in $X \times X$. Tkachuk and Burke showed in [8] that a large discrete set can be found in $X \times X$ if X is a Lindelöf Σ-space.

Observe first that some restrictions on X are necessary for existence of a large discrete subset in $X \times X$ because under CH there exists a strong L-space X (see [20, Theorem 4.2] and [3, (1.6.12)]); therefore $s(X^2) \leq hl(X^2) = \omega$ but $d(X) > \omega$ so there is no large discrete subset in $X \times X$.

In [8] it was also proved that if X is a pseudocompact space of weight $\leq \omega_1$ then it is possible to find a large discrete set in $X \times X$. The following proposition generalizes this result.

Proposition 3.9. *If X is a non-Lindelöf space and $d(X) \leq \omega_1$ then there exists a discrete subset in $D \subset X \times X$ such that $|D| = d(X)$.*

Proof. Since every infinite subset of $X \times X$ contains an infinite discrete subset, the proof is trivial if $d(X) = \omega$. If $d(X) = \omega_1$ then $hd(X) > \omega$ so we can find a faithfully indexed set $L = \{x_\alpha : \alpha < \omega_1\} \subset X$ such that $\{x_\beta : \beta < \alpha\}$ is closed in L for any $\alpha < \omega_1$. It follows from $hl(X) \geq l(X) > \omega$ that there exists a faithfully indexed set $R = \{y_\alpha : \alpha < \omega_1\} \subset X$ such that $\{y_\beta : \beta < \alpha\}$ is open in R for any $\alpha < \omega_1$. It is immediate that the set $D = \{(x_\alpha, y_\alpha) : \alpha < \omega_1\} \subset X \times X$ is discrete and $|D| = \omega_1$. □

4. Squares of spaces with a countable network

We already mentioned that van Douwen's example of countable maximal space shows that in the class of spaces with a countable network we cannot expect metrizability, first countability, Čech-completeness and many other properties to be discretely reflexive. However, van Douwen's example does not discard the possibility for analyticity and σ-compactness to be reflexive in spaces with a countable network. The respective questions were asked in [8].

Recall that a family $\mathcal{F}$ of subsets of a space X is *T_2-separating* for a set $Y \subset X$ if for any distinct $x, y \in Y$ there are disjoint $F, G \in \mathcal{F}$ such that $x \in F$ and $y \in G$. We will need the following easy observation.

Lemma 4.1. *If X is a space and $Y \subset X$ is infinite then no finite family $\mathcal{F}$ of subsets of X can be T_2-separating for Y.*

It was established in [8] that for any Lindelöf p-space X, there is a discrete set $D \subset X \times X$ such that $\Delta_X \subset \overline{D}$. Here $\Delta_X = \{(x, x) : x \in X\} \subset X \times X$ is the diagonal of the space X. This result shows that quite a few properties are discretely reflexive in $X \times X$ for a Lindelöf p-space X.

Unfortunately, there is no hope to prove the same for spaces with a countable network because it was shown in [8] that under CH there exists a countable crowded space X such that all discrete subsets of $X \times X$ are closed in X. In particular, the diagonal is not contained in the closure of discrete subset of X. However, we will see that the closure of a discrete set can still contain an essential part of the diagonal of a space with a countable network.

Definition 4.2. *If X is a space and $Y \subset X$, say that a family $\mathcal{N}$ of non-empty subsets of X is a π-network for Y if for any set $U \in \tau(X)$ such that $U \cap Y \neq \emptyset$ we can find a set $N \in \mathcal{N}$ with $N \subset U$.*

Theorem 4.3. *Suppose that X is a space, $Y \subset X$ and there exists a countable π-network $\mathcal{F}$ in X for the subspace Y such that every $F \in \mathcal{F}$ is infinite. Then it is possible to find a discrete set $D \subset (X \times X) \backslash \Delta_X$ such that $\Delta_Y \subset \overline{D}$.*

Proof. Let $\{F_n : n \in \omega\}$ be an enumeration of $\mathcal{F}$. Since F_0 is infinite, we can pick distinct points $a_0, b_0 \in F_0$; take disjoint sets $U_0, V_0 \in \tau(X)$ such that $a_0 \in U_0$ and $b_0 \in V_0$. Proceeding inductively, assume that $n \in \omega$ and we have points $a_0, b_0, \ldots, a_n, b_n \in X$ and sets $U_0, V_0, \ldots, U_n, V_n \in \tau(X)$ with the following properties:

(1) $a_i \in U_i \cap F_i$ and $b_i \in V_i \cap F_i$ for all $i \leq n$;
(2) $U_i \cap V_i = \emptyset$ for every $i \leq n$;
(3) $(a_{i+1}, b_{i+1}) \notin \bigcup\{U_j \times V_j : j \leq i\}$ for all $i < n$.

Observe that $(F_{n+1} \times F_{n+1}) \backslash \Delta_X$ is not contained in $\bigcup\{U_j \times V_j : j \leq n\}$ for otherwise the finite family $\{U_j, V_j : j \leq n\}$ will be T_2-separating for the infinite set F_{n+1} which is a contradiction with Lemma 4.1. Therefore we can pick distinct $a_{n+1}, b_{n+1} \in F_{n+1}$ such that $(a_{n+1}, b_{n+1}) \notin \bigcup\{U_j \times V_j : j \leq n\}$. Take disjoint sets $U_{n+1}, V_{n+1} \in \tau(X)$ for which $a_{n+1} \in U_{n+1}$ and $b_{n+1} \in V_{n+1}$; then the conditions (1)–(3) are satisfied if we replace n with $n+1$. Therefore our inductive procedure can be continued to obtain the sets $\{a_i : i \in \omega\}$ and $\{b_i : i \in \omega\}$ together with the families $\{U_i : i \in \omega\}$ and $\{V_i : i \in \omega\}$ for which the properties (1)–(3) hold for all $n \in \omega$.

The set $D = \{(a_n, b_n) : n \in \omega\} \subset (X \times X) \backslash \Delta_X$ is discrete because for each $n \in \omega$ we have the inclusion $(U_n \times V_n) \cap D \subset \{(a_i, b_i) : i \leq n\}$ by the property (3). To see that $\Delta_Y \subset \overline{D}$ take any point $x \in Y$ and an arbitrary open neighborhood

G of the point (x,x). There exists $W \in \tau(x,X)$ such that $W \times W \subset G$. Since $x \in W \cap Y$, we can pick a set $F \in \mathcal{F}$ with $F \subset W$; there exists a number $n \in \omega$ such that $F = F_n$. The property (1) shows that $(a_n, b_n) \in F_n \times F_n \subset W \times W \subset G$ and therefore $(a_n, b_n) \in D \cap G$. Thus $(x,x) \in \overline{D}$ for any $x \in Y$. □

Corollary 4.4. *If X is a space with a countable network then there exists a discrete set $D \subset (X \times X)\backslash\Delta_X$ such that $\Delta_X \backslash \overline{D}$ is countable.*

Proof. Let $\mathcal{F}$ be a countable network in the space X. If $\mathcal{F}_0 = \{F \in \mathcal{F} : F$ is finite$\}$ then the set $E = \bigcup \mathcal{F}_0$ is countable. If $E = X$ then we are done; otherwise take a point $y \in Y = X\backslash E$ and observe that any element of $\mathcal{F}$ containing y is infinite. Since $\mathcal{F}\backslash\mathcal{F}_0$ is a network at every point of Y, it is a countable π-network of infinite sets for Y. By Theorem 4.3 we can find a discrete set $D \subset (X \times X)\backslash\Delta_X$ such that $\Delta_Y \subset \overline{D}$. This implies that $\Delta_X \backslash \overline{D} \subset E \times E$ is a countable set so D is as promised. □

Corollary 4.5. *If X is a space with a countable π-base then there exists a discrete set $D \subset X \times X$ such that $\Delta_X \subset \overline{D}$.*

Proof. Denote by I the set of isolated points of X. The set $Y = X\backslash\overline{I}$ is open in X so we can fix a countable π-base $\mathcal{U}$ in Y. The set Y being crowded, all elements of $\mathcal{U}$ are infinite so Theorem 4.3 is applicable to find a discrete $D_0 \subset (Y \times Y)\backslash\Delta_Y$ such that $\Delta_Y \subset \overline{D}_0$. Then $D = D_0 \cup \Delta_I$ is a discrete subset of $X \times X$ such that $\Delta_X \subset \overline{D}$. □

Theorem 4.6. *Assume that a topological property $\mathcal{P}$ is closed-hereditary, X is a space with $\pi w(X) \le \omega$ and $\overline{D}$ has $\mathcal{P}$ for any discrete $D \subset X \times X$. Then X has $\mathcal{P}$.*

Proof. Apply Corollary 4.5 to find a discrete set $D \subset X \times X$ such that $\Delta_X \subset \overline{D}$. Since Δ_X is closed in $\overline{D}$, it must have $\mathcal{P}$; therefore X has $\mathcal{P}$ as well being homeomorphic to Δ_X. □

Corollary 4.7. *Assume that a topological property $\mathcal{P}$ is closed-hereditary, X is a first countable separable space and $\overline{D}$ has $\mathcal{P}$ for any discrete $D \subset X \times X$. Then X has $\mathcal{P}$.*

Proof. First countable separable spaces have a countable π-base; Theorem 4.6 does the rest. □

It follows from Theorem 4.6 that if a space X has countable π-weight, then all properties that are closed-hereditary and stable under finite products are discretely reflexive in $X \times X$. This includes, but is not limited to zero-dimensionality, countability, σ-compactness, Čech-completeness, analyticity.

The following example shows that the situation is different if we consider discrete reflexivity of properties in X.

Example 4.8. It follows from a result of van Douwen (see [9, Theorem A]) that there exists a point $p \in \beta\mathbb{R}\backslash\mathbb{R}$ which is *remote* for $\mathbb{R}$, i.e., p is not in the closure of any nowhere dense subset of $\mathbb{R}$. Observe that the space $X = \mathbb{R}\cup\{p\}$ has countable π-weight while $\overline{D}$ is second countable and Čech-complete for any discrete $D \subset X$. However, X is not a k-space so there is a long list of nice properties including (but not limited to) first countability, countable weight, Fréchet–Urysohn property, and Čech-completeness that are not discretely reflexive in X.

Definition 4.9. *Say that a topological property $\mathcal{P}$ satisfies the countable sum condition in a class $\mathcal{C}$ if for any space $X \in \mathcal{C}$, if $X = \bigcup_{n\in\omega} X_n$ where every X_n is closed in X and has $\mathcal{P}$ then X has $\mathcal{P}$.*

Proposition 4.10. *Suppose that a property $\mathcal{P}$ is closed-hereditary and X is a space with a countable network such that $X \times X$ is discretely $\mathcal{P}$.*
(a) if the property $\mathcal{P}$ satisfies the countable sum condition in spaces with a countable network, then X has $\mathcal{P}$.
(b) if X is regular and $\mathcal{P}$ satisfies the countable sum condition in regular spaces with a countable network, then X has $\mathcal{P}$.

Proof. We will prove (a) and (b) simultaneously. If $z \in X \times X$ then $D = \{z\}$ is discrete and $D = \overline{D}$ so all one-point subspaces of $X \times X$ trivially have the property $\mathcal{P}$. Apply Theorem 4.3 to find a discrete set $D \subset X\times X$ such that $\Delta_X\backslash\overline{D}$ is countable. The set $F = \overline{D} \cap \Delta_X$ is closed in $\overline{D}$ and hence has the property $\mathcal{P}$. Therefore $\Delta_X = \bigcup\mathcal{F}$ where the family $\mathcal{F} = \{F\} \cup \{\{(x,x)\} : (x,x) \in \Delta_X\backslash F\}$ is countable and all elements of $\mathcal{F}$ have $\mathcal{P}$. Since both regularity and countable network weight are hereditary and finitely productive, in both cases (a) and (b) the set Δ_X has $\mathcal{P}$. Finally observe that Δ_X is homeomorphic to X so the space X also has the property $\mathcal{P}$. □

The following corollary answers Problem 3.3 from [8].

Corollary 4.11. *If X is a regular space with a countable network and $\overline{D}$ is zero-dimensional for any discrete $D \subset X \times X$, then X is zero-dimensional.*

Proof. Although the small inductive dimension is not preserved by countable unions of closed sets in general spaces, a Lindelöf space X is zero-dimensional if and only if $\dim(X) = 0$; besides, the dimension dim satisfies the countable sum condition in normal spaces. Every regular space of countable network weight is Lindelöf and hence normal; therefore zero-dimensionality satisfies the countable sum condition in regular spaces with a countable network. As a consequence, Proposition 4.10(b) can be applied to conclude that the space X is zero-dimensional whenever $X \times X$ is discretely zero-dimensional. □

The next statement settles Problem 3.8 from [8].

Corollary 4.12. *If X is a space with a countable network and $\overline{D}$ is σ-compact for any discrete $D \subset X \times X$ then X is σ-compact.*

PROOF. It suffices to observe that σ-compactness satisfies all the premises of Proposition 4.10(a). □

The following corollary solves Problem 3.10 from [8].

Corollary 4.13. *If X is a space with a countable network and $\overline{D}$ is analytic for any discrete $D \subset X \times X$ then X is analytic.*

PROOF. Clearly, analyticity satisfies the premises of Proposition 4.10(a). □

It was asked in [8] whether X must be countable provided that $X \times X$ is discretely countable. Our methods enable us to give a positive answer for Lindelöf Σ-spaces.

Proposition 4.14. *If X is a regular Lindelöf Σ-space and $\overline{D}$ has a countable network for any discrete $D \subset X \times X$ then $nw(X) \leq \omega$.*

PROOF. By [1, Proposition 2.1], hereditary Lindelöf number is discretely reflexive in any space so $X \times X$ is hereditarily Lindelöf. Therefore X is Lindelöf and its diagonal Δ_X is a G_δ-subset of the space $X \times X$. Applying Theorem 2.1.8 of [3] we convince ourselves that $iw(X) \leq \omega$ and hence $nw(X) \leq \omega$ by stability of X (see [4, Theorem II.6.21]). □

Corollary 4.15. *If X is a regular Lindelöf Σ-space and $\overline{D}$ is countable for any discrete $D \subset X \times X$ then X is countable.*

PROOF. By Proposition 4.14, the space X must have a countable network. The property of being countable satisfies the premises of Proposition 4.10(a) so the space X must be countable. □

The example that follows provides a consistent answer to Problem 3.4 and Problem 3.6 of [8].

Example 4.16. Under the Continuum Hypothesis there exists a strong L-space X (see [20, Theorem 4.2] and [3, (1.6.12)]). Passing to a subspace if necessary, we can assume, without loss of generality, that $X = \{x_\alpha : \alpha < \omega_1\}$ and the set $X_\alpha = \{x_\beta : \beta < \alpha\}$ is closed in X for any $\alpha < \omega_1$. The space X being left-separated, every compact subspace $K \subset X$ is scattered by [14, Theorem 1] and hence has a countable dense set E of isolated points. There is $\alpha < \omega_1$ such that $E \subset X_\alpha$; since X_α is closed in X, we have $K = \overline{E} \subset X_\alpha$ and therefore K is countable. This shows that space X is not σ-compact.

Now, if $D \subset X \times X$ is a discrete set then D is countable and hence there exists $\alpha < \omega_1$ such that $D \subset X_\alpha \times X_\alpha$. The set $X_\alpha \times X_\alpha$ being closed, we have $\overline{D} \subset X_\alpha \times X_\alpha$ so $|\overline{D}| = \omega$ for any discrete $D \subset X \times X$ while X is not σ-compact.

Since convergence properties are not discretely reflexive in $X \times X$ for spaces X with a countable network, the following result might be of interest.

Proposition 4.17. *If $nw(X) \leq \omega$ then there exists a discrete subspace $D \subset X^\omega$ such that X^ω is homeomorphic to a closed subspace of $\overline{D}$.*

PROOF. If X is a singleton then X^ω is discrete so there is nothing to prove. If $|X| > 1$ then it is an easy exercise to check that

($*$) for any countable $A \subset X^\omega$ there exists a closed set $F \subset X^\omega \backslash A$ which is homeomorphic to X^ω.

Apply Theorem 4.3 to the space $Y = X^\omega$ to find a discrete set $D \subset Y \times Y$ such that $B = \Delta_Y \backslash \overline{D}$ is countable. Since Δ_Y is homeomorphic to Y, the property ($*$) shows that some closed subset F of $\Delta_Y \backslash B$ is homeomorphic to Y. After we identify $Y \times Y$ with X^ω, the set F will be the promised closed copy of X^ω contained in $\overline{D}$. □

Corollary 4.18. *Suppose that $\mathcal{P}$ is a topological property invariant under closed subspaces and $nw(X) \leq \omega$. Then the space X^ω has $\mathcal{P}$ if and only if $\overline{D}$ has $\mathcal{P}$ for any discrete $D \subset X^\omega$. In particular any property from the list {first countability, Fréchet–Urysohn property, sequentiality, Čech-completeness} is discretely reflexive in X^ω.*

5. Open problems

Discrete reflexivity in the squares of spaces turned out to be an interesting topic with a potential to provide new information about such classical properties as σ-compactness and analyticity even in spaces with a countable network. The following list of open questions shows that this topic is still far from being exhausted.

Question 5.1. *Suppose that X is a Lindelöf Σ-space. Is it true that there exists a discrete set $D \subset X \times X$ such that $\Delta_X \subset \overline{D} \cup S$ for some σ-compact set $S \subset X \times X$?*

Question 5.2. *Suppose that X is a σ-compact space. Does there exist a discrete $D \subset X \times X$ such that the projection of D onto the first coordinate is dense in X?*

Question 5.3. *Suppose that X is a compact space and $\overline{D}$ is hereditarily normal for any discrete $D \subset X^3$. Must X be metrizable?*

Question 5.4. *Suppose that X is a countably compact space and $\chi(\overline{D}) \leq \omega$ for any discrete $D \subset X \times X$. Must X be first countable?*

Question 5.5. *Suppose that X is a countably compact space and $t(\overline{D}) \leq \omega$ for any discrete $D \subset X \times X$. Is it true that $t(X) \leq \omega$?*

Question 5.6. *Suppose that X is a countably compact space and $\overline{D}$ is a Fréchet–Urysohn space for any discrete $D \subset X \times X$. Must X be Fréchet–Urysohn?*

Question 5.7. *Suppose that X is a countably compact space and $\overline{D}$ is a sequential space for any discrete $D \subset X \times X$. Must X be sequential?*

Question 5.8. *Suppose that X is a Lindelöf Σ-space such that $(X \times X)\backslash\Delta$ is discretely σ-compact. Is true in ZFC that $nw(X) \leq \omega$?*

Question 5.9. *Suppose that X is a pseudocompact space such that $(X \times X)\backslash\Delta$ is discretely σ-compact. Must X be metrizable?*

Question 5.10. *Suppose that X is a Lindelöf Σ-space such that $\chi(\overline{D}) \leq \omega$ for any discrete $D \subset X^\omega$. Must X be first countable?*

Question 5.11. *Suppose that $\pi w(X) \leq \omega$ and $\overline{D}$ is Lindelöf for any discrete $D \subset X \times X$. Then X is Lindelöf but must $X \times X$ be Lindelöf?*

Question 5.12. *Suppose that $\pi w(X) \leq \omega$ and $\overline{D}$ is Fréchet–Urysohn for any discrete $D \subset X \times X$. Then X is Fréchet–Urysohn, but must $X \times X$ be Fréchet–Urysohn?*

Question 5.13. *Suppose that $\pi w(X) \leq \omega$ and $\overline{D}$ is sequential for any discrete $D \subset X \times X$. Then X is sequential, but must $X \times X$ be sequential?*

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