My paper accepted in Mathematica Slovaca

2 messages

Vladimir Tkachuk <vvtmdf@gmail.com>  Mon, Nov 14, 2016 at 8:03 PM
To: Mathematica Slovaca <maslo@mat.savba.sk>

Dear professor Surova:

My paper entitled "Lindelof P-spaces need not be Sokolov" was accepted for publication in Mathematica Slovaca in March 2015. Could you, please tell me when, approximately it could be published?

All the best

Vladimir Tkachuk

Mathematica Slovaca <maslo@mat.savba.sk>  Tue, Nov 15, 2016 at 12:41 AM
To: vvtmdf@gmail.com

Dear Professor Tkachuk,

Your paper "Lindelof P-spaces need not be Sokolov" will be published in the Volume 66/2016.

Best regards

A. Surova
Editorial Office
Math. Slovaca

Dear professor Surova:

My paper entitled "Lindelof P-spaces need not be Sokolov" was accepted for publication in Mathematica Slovaca in March 2015. Could you, please tell me when, approximately it could be published?

All the best

Vladimir Tkachuk
LINDELÖF P-SPACES NEED NOT BE SOKOLOV

V.V. TKACHUK

Abstract. We show that for every Lindelöf $P$-space a weaker version of the Sokolov property holds. Besides, if $K$ is a scattered Eberlein compact space and $X$ is obtained from $K$ by declaring open all $G_δ$-subsets of $K$, then $X$ is monotonically Sokolov. The proof of this statement uses the fact that every Lindelöf subspace of a scattered Eberlein compact space must be $σ$-compact; this result seems to be interesting in itself. We also give an example of a Lindelöf $P$-space $X$ such that $C_p(X)$ has uncountable extent. In particular, neither $X$ nor $C_p(X)$ has the Sokolov property.

1. Introduction.

In this paper we deal with the Sokolov property in Lindelöf $P$-spaces. Recall that $X$ is a Sokolov space (or has the Sokolov property) if for every sequence $\{F_n : n ∈ \mathbb{N}\}$, where each $F_n$ is a closed subset of $X^n$, there exists a continuous mapping $f : X → X$ such that $nw(f(X)) ≤ ω$ and $f^n(F_n) ⊂ F_n$ for every $n ∈ \mathbb{N}$. This class of spaces was introduced by Sokolov in [7] (under a different name); it was established in [7] that every Corson compact space is Sokolov; besides, $X$ is Sokolov if and only if $C_p(X)$ is Sokolov and if $X$ is a compact Sokolov space, then every iterated function space $C_{p,n}(X)$ is Lindelöf. In the paper [10] the class of Sokolov spaces was studied systematically and it was proved, among other things, that every Sokolov space is collectionwise normal, $ω$-stable, $ω$-monolithic and has countable extent.

In the paper [5] monotonically Sokolov spaces were introduced and it was proved that every monotonically Sokolov space is monotonically $ω$-monolithic and Sokolov. It was also established that if $X$ and $C_p(X)$ are Lindelöf $Σ$-spaces, then they are both monotonically Sokolov; this answered a question from [10] in a stronger form.

We show that every Lindelöf $P$-space $X$ has a weaker version of the Sokolov property, namely, for any countable family $F$ of closed subspaces of $X$ there exists a retraction $r : X → X$ such that the set $r(X)$ is countable and we have

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the inclusion \( r(F) \subset F \) for any \( F \in \mathcal{F} \). A wide class of Lindelöf \( P \)-spaces can be obtained from compact scattered spaces by declaring open their \( G_\delta \)-sets; this operation is called \( \omega \)-modification. We show that the \( \omega \)-modification \( X \) of a scattered Eberlein compact space \( K \) must be monotonically Sokolov and, in particular, the extent of the space \( C_p(X) \) is countable. The proof of this fact uses the following property of scattered Eberlein compact spaces which seems to be interesting in itself: if \( K \) is a scattered Eberlein compact space, then every Lindelöf subspace of \( K \) is \( \sigma \)-compact.

If \( X \) is a Sokolov space, then \( C_p(X) \) is also Sokolov and hence we have the equality \( \text{ext}(C_p(X)) = \omega \). In the case of a Lindelöf \( P \)-space \( X \), the set \( C_p(X, [0, 1]) \) is countably compact and, in particular, \( \text{ext}(C_p(X, [0, 1])) = \omega \); this gave the author hope that \( \text{ext}(C_p(X)) = \omega \) for any Lindelöf \( P \)-space \( X \). However, this turned out to be false: we give the respective example answering in the negative Problem 3.12 from [10]. We also show that if \( X \) is a Lindelöf \( P \)-space, then a compact subspace of \( C_p(X) \) need not be Sokolov answering Problem 3.13 from [10].

2. Notation and terminology.

All spaces are assumed to be Tychonoff. Given a space \( X \), the family \( \tau(X) \) is its topology and \( \tau(x, X) = \{ U \in \tau(X) : x \in U \} \) for any point \( x \in X \). The set \( \mathbb{R} \) is the real line with its usual topology and \( \mathbb{N} = \{1, 2, \ldots \} \subset \mathbb{R} \). We denote by \( \mathbb{D} \) the set \( \{0, 1\} \) with the discrete topology. A space \( X \) is scattered if every non-empty subspace of \( X \) has an isolated point. If \( A \) is a set then \( \Sigma(A) = \{x \in \mathbb{R}^A : \text{for any } \varepsilon > 0 \text{ the set } \{a \in A : |x(a)| \geq \varepsilon \} \text{ is finite} \} \) is called the \( \Sigma_\varepsilon \)-product of \( \mathbb{R}^A \). A compact space \( K \) is Eberlein compact if \( K \) embeds in \( \Sigma(A) \) for some \( A \). We say that \( X \) is a \( P \)-space if every \( G_\delta \)-subset of \( X \) is open.

If \( \varphi : X \to Y \) is a map then \( \varphi^n : X^n \to Y^n \) is defined by the formula \( \varphi^n(x) = (\varphi(x_1), \ldots, \varphi(x_n)) \) for any point \( x = (x_1, \ldots, x_n) \in X^n \) and \( n \in \mathbb{N} \). Given a set \( A \) in a space \( X \), say that a family \( \mathcal{N} \) of subsets of \( X \) is an external network of \( A \) in \( X \) if for any \( x \in A \) and \( U \in \tau(x, X) \) there exists \( N \in \mathcal{N} \) such that \( x \in N \subset U \). An external network of \( X \) in \( X \) is called a network in \( X \). The cardinal \( \text{nw}(X) = \min\{|N| : N \text{ is a network of } X\} \) is called the network weight of \( X \) and \( \text{ext}(X) = \sup\{|D| : D \text{ is a closed discrete subset of } X\} \) is the extent of the space \( X \). If \( \kappa \) is an infinite cardinal and \( X \) is a space then \( (X)_\kappa \) is the set \( X \) with the topology generated by all \( G_\kappa \)-subsets of \( X \). The space \( (X)_\kappa \) is called the \( \kappa \)-modification of \( X \).

Given an infinite cardinal \( \kappa \) say that a space \( X \) is monotonically \( \kappa \)-monolithic if, to any set \( A \subset X \) with \( |A| \leq \kappa \), we can assign an external network \( \mathcal{O}(A) \) to the set \( A \) in such a way that the following conditions are satisfied:

(a) \( |\mathcal{O}(A)| \leq \max\{|A|, \omega\} \);
(b) if \( A \subset B \subset X \) and \( |B| \leq \kappa \) then \( \mathcal{O}(A) \subset \mathcal{O}(B) \);
(c) if $\lambda \leq \kappa$ is a cardinal and we have a family $\{A_\alpha : \alpha < \lambda\} \subset [X]^{<\kappa}$ such that $\alpha < \beta < \lambda$ implies $A_\alpha \subset A_\beta$ then $O(\bigcup_{\beta < \lambda} A_\alpha) = \bigcup_{\alpha < \lambda} O(A_\alpha)$.

Suppose that $X$ is a set, a family $A \subset \exp(X)$ is closed under countable increasing unions, $B \subset \exp(Y)$ and we have a map $\varphi : A \to B$. We say that $\varphi$ is $\omega$-monotone if

1. $\varphi(A)$ is countable whenever $A \in A$ is countable;
2. if $A \subset B$ and $A, B \in A$, then $\varphi(A) \subset \varphi(B)$;
3. if $\{A_n : n \in \omega\} \subset A$ and $A_n \subset A_{n+1}$ for any $n \in \omega$ then we have the equality $\varphi(\bigcup_{n \in \omega} A_n) = \bigcup_{n \in \omega} \varphi(A_n)$.

Say that a space $X$ is monotonically Sokolov if we can assign to any countable family $F$ of closed subsets of $X$ a continuous retraction $r_F : X \to X$ and a countable external network $N(F)$ for $r_F(X)$ in $X$ such that $r_F(F) \subset F$ for each $F \in F$ and the assignment $N$ is $\omega$-monotone.

For any spaces $X$ and $Y$ the set $C(X, Y)$ consists of continuous functions from $X$ to $Y$; if it has the topology induced from $Y^X$ then the respective space is denoted by $C_p(X, Y)$. We write $C_p(X)$ instead of $C_p(X, \mathbb{R})$. The rest of our topological notation is standard and follows the book [3]. For unreferenced notions of $C_p$-theory, see the books [2] and [11].

3. Retractions in Lindelöf $P$-spaces

We will show that there are rich families of retractions in every Lindelöf $P$-space $X$ and, in particular, a weaker form of Sokolov property holds in $X$. We will see that a general Lindelöf $P$-space is not necessarily Sokolov. However, if it is obtained as an $\omega$-modification of a nice scattered compact space, then it can be even monotonically Sokolov.

3.1. Definition. Suppose that $X$ is a space and $F$ is a family of subsets of $X$. Say that a set $A \subset X$ is dense with respect to $F$ if $A \cap \bigcap F' \neq \emptyset$ for any family $F' \subset F$ such that $\bigcap F' \neq \emptyset$. The family $F$ will be called separable if some countable subset of $X$ is dense with respect to $F$.

3.2. Observation. Even for a second countable space $X$ we cannot hope that every countable family of closed subsets of $X$ is separable. For example, the family $Q$ of closed rational intervals in $\mathbb{R}$ is countable, consists of compact subsets of $\mathbb{R}$ but fails to be separable because every $x \in \mathbb{R}$ is the intersection of a subfamily of $Q$. However, we will see that the situation in Lindelöf $P$-spaces is different.

3.3. Proposition. If $X$ is a Lindelöf $P$-space, then any countable family $F$ of closed subsets of $X$ is separable.

Proof. For every point $x \in X$ let $F_x = \{F \in F : x \in F\}$ and choose a clopen set $U_x$ such that $x \in U_x$ and $U_x \cap F = \emptyset$ for any $F \in F \setminus F_x$. There exists a
countable set \( A \subset X \) for which \( X = \bigcup \{ U_x : x \in A \} \); we claim that \( A \) is as required. Indeed, take a subfamily \( \mathcal{F}' \subset F \) such that \( \bigcap \mathcal{F}' \neq \emptyset \); if \( y \in \bigcap \mathcal{F}' \) then there is a point \( x \in A \) such that \( y \in U_x \). For every \( F \in \mathcal{F}' \) it follows from \( F \cap U_x \neq \emptyset \) that \( x \in F \) so \( x \in A \cap \bigcap \mathcal{F}' \), i.e., \( A \) is dense with respect to \( F \). \( \square 
\)

3.4. Theorem. Suppose that \( X \) is a Lindel"of \( P \)-space, \( A \subset X \) is a countable set and \( F \) is a countable family of closed subsets of \( X \). Then the following conditions are equivalent:

(a) there exists a retraction \( r : X \to A \) such that \( r(F) \subset F \) for every \( F \in F \);

(b) the set \( A \) is dense with respect to \( F \).

Proof. Suppose that \( r : X \to A \) is a retraction as in (a) and \( \mathcal{F}' \subset F \) is a subfamily such that \( Q = \bigcap \mathcal{F}' \neq \emptyset \). If \( A \cap Q = \emptyset \) then take a point \( x \in Q \) and observe that \( r(x) \notin Q \) so there exists a set \( F \in \mathcal{F}' \) such that \( r(x) \notin F \) and hence \( r(F) \not\subset F \); this contradiction shows that (a)\( \implies \) (b).

Now, if (b) holds, then for every \( x \in X \) apply the \( P \)-property of \( X \) to find a clopen set \( U_x \) such that \( x \in U_x \) and \( F \cap U_x = \emptyset \) implies \( x \in F \) for any \( F \in F \). Making the sets \( U_x \) smaller if necessary, we can assume that the family \( \{ U_x : x \in A \} \) is disjoint. Since \( X \) is Lindel"of, we can find a disjoint clopen cover \( \{ W_n : n \in \omega \} \) of the set \( E = X \setminus \bigcup \{ U_x : x \in A \} \) such that every \( W_n \) is contained in the set \( U_y \) for some \( y \in E \).

If \( z \in U_x \) for some \( x \in A \) then let \( r(z) = x \). If \( z \in W_n \) then \( W_n \subset U_y \) for some \( y \in E \); consider the family \( \mathcal{F}' = \{ F \in F : F \cap W_n \neq \emptyset \} \). It follows from the choice of \( U_y \) that \( y \in \bigcap \mathcal{F}' \) so \( \bigcap \mathcal{F}' \neq \emptyset \) and hence there exists \( a \in A \cap \bigcap \mathcal{F}' \); let \( r(z) = a \).

The map \( r : X \to X \) is continuous being constant on every element of the clopen cover \( \mathcal{U} = \{ U_x : x \in A \} \cup \{ W_n : n \in \omega \} \) of the space \( X \). It is clear that \( r(a) = a \) for any \( a \in A \) so \( r : X \to A \) is a continuous retraction and \( r(F) \subset F \) for any \( F \in F \) by our choice of the cover \( \mathcal{U} \); this proves that (b)\( \implies \) (a). \( \square 
\)

It was established in [7] that a space \( X \) is Sokolov if and only if it has the following property:

(\( f \)) if \( F_{nm} \) is a closed subset of \( X^n \) for all \( n, m \in \mathbb{N} \), then there is a continuous map \( f : X \to X \) such that \( n\omega(f(X)) \leq \omega \) and we have the inclusion \( f^n(F_{nm}) \subset F_{nm} \) for all \( n, m \in \mathbb{N} \).

Therefore, the following corollary shows that for any Lindel"of \( P \)-space a weaker version of the Sokolov property holds.

3.5. Corollary. If \( X \) is a Lindel"of \( P \)-space and \( F \) is a countable family of closed subsets of \( X \) then there exists a countable set \( A \subset X \) and a continuous retraction \( r : X \to A \) such that \( r(F) \subset F \) for any \( F \in F \).

Proof. Apply Proposition 3.3 and Theorem 3.4. \( \square 
\)
3.6. Proposition. Given a Lindelöf P-space \( X \) suppose that for any countable family \( \mathcal{F} \) of closed subsets of \( X \) we can find a countable family \( \mathcal{N}(\mathcal{F}) \) of subsets of \( X \) and a countable set \( A(\mathcal{F}) \subset X \) such that \( A(\mathcal{F}) \) is dense with respect to \( \mathcal{F} \), the assignment \( \mathcal{F} \to \mathcal{N}(\mathcal{F}) \) is \( \omega \)-monotone and \( \{ \{ x \in A(\mathcal{F}) \} \subset \mathcal{N}(\mathcal{F}) \). Then \( X \) is monotonically Sokolov.

Proof. By Corollary 3.5 there exists a continuous retraction \( r_\mathcal{F} : X \to A(\mathcal{F}) \); if \( x \in A(\mathcal{F}) \), then \( \{ \{ x \} \} \) is an external network at \( x \) so it follows from the inclusion \( \{ \{ x \in A(\mathcal{F}) \} \subset \mathcal{N}(\mathcal{F}) \) that \( \mathcal{N}(\mathcal{F}) \) is an external network at every point of \( A(\mathcal{F}) \), i.e., \( X \) is monotonically Sokolov. \( \square \)

For any space \( X \) let \( I(X) \) be the set of isolated points of \( X \). If \( X \) is a scattered space then let \( X^{(0)} = X \); if \( \alpha \) is an ordinal and we have a set \( X^{(\alpha)} \subset X \) then \( X^{(\alpha+1)} = X^{(\alpha)} \setminus I(X^{(\alpha)}) \). If \( \beta \) is a limit ordinal and we have \( X^{(\alpha)} \) for any \( \alpha < \beta \), then \( X^{(\beta)} = \bigcap \{ X^{(\alpha)} : \alpha < \beta \} \). The first ordinal \( \alpha \) such that \( X^{(\alpha)} = \emptyset \) is called the dispersion index (or the height) of a scattered space \( X \); it is denoted by \( di(X) \).

3.7. Theorem. Suppose that \( K \) is a scattered Eberlein compact. If a subspace \( L \subset K \) is Lindelöf then it is \( \sigma \)-compact.

Proof. Let us temporarily call a scattered Eberlein compact space \( X \) adequate if every Lindelöf subspace of \( X \) is \( \sigma \)-compact. Our goal is to prove that all scattered Eberlein compact spaces are adequate. Observe first that

1. any closed subspace of an adequate space is adequate;
2. if \( X = \bigcup_{n \in \omega} X_n \) and every \( X_n \subset X \) is adequate then \( X \) is adequate.

Apply [6, Proposition 8] and [1, Corollary 2] to find a cardinal \( \kappa \) such that \( K \) embeds in the \( \kappa \)-product \( S = \{ x \in D_\kappa : |x^{-1}(1)| < \omega \} \subset D_\kappa \); we can assume that \( K \subset S \). Note that \( S_n = \{ x \in S : |x^{-1}(1)| \leq n \} \) is a compact subset of \( S \) for any \( n \in \omega \) and \( S = \bigcup_{n \in \omega} S_n \). It is easy to see that \( I(S_n) = S_n \setminus S_{n-1} \) and hence \( S_n \setminus I(S_n) = S_{n-1} \), so a straightforward induction shows that the dispersion index of \( S_n \) is equal to \( n + 1 \) for every \( n \in \omega \). If \( K_n = K \cap S_n \) then \( K = \bigcup_{n \in \omega} K_n \); by the properties (1)-(2) it is sufficient to prove that every \( K_n \) is adequate. The observation above implies that \( di(K_n) \leq di(S_n) = n + 1 \) for each \( n \in \omega \) and therefore the following statement is all we need to finish the proof.

\((\ast)\) Any Eberlein compact space of finite dispersion index is adequate.

We will prove \((\ast)\) by induction on dispersion index. If \( di(X) = 1 \), then the space \( X \) is finite so \((\ast)\) holds. Suppose that \( n \in \mathbb{N} \) and we proved \((\ast)\) for the spaces of dispersion index \( \leq n \). Take an Eberlein compact space \( X \) such that \( di(X) = n + 1 \). The set \( X^{(n)} \) is finite, say \( X^{(n)} = \{ z_1, \ldots, z_k \} \). It is easy to find disjoint clopen sets \( U_1, \ldots, U_k \) such that \( X = U_1 \cup \ldots \cup U_k \) and \( z_i \in U_i \) for any \( i \leq k \). It follows from (3) that it suffices to prove \((\ast)\) for every \( U_i \) so we can assume, without loss of generality, that \( |X^{(n)}| = 1 \). Let \( p \in X \) be the point such that \( X^{(n)} = \{ p \} \). Then \( X^{(n-1)} = D \cup \{ p \} \) where \( p \notin D \) and \( D \) is a discrete set.
Take an arbitrary Lindelöf subspace $L \subseteq X$. If $p \notin L$ then we can find a $G_δ$-set $P$ such that $p \in P \subseteq X\setminus L$ so $X\setminus P = \bigcup_{n\in\omega} F_n$ where $F_n$ is compact and $di(F_n) \le n$ so $F_n$ is adequate for any $n \in \omega$. This implies that $L_n = L \cap F_n$ is $σ$-compact for each $n$ and therefore $L = \bigcup_{n\in\omega} L_n$ is $σ$-compact.

Now assume that $p \in L$ and hence the set $Q = L \cap X^{(n-1)}$ is compact. Let $ϕ$ be the map on $X$ defined by collapsing the set $X^{(n-1)}$ to a point $q$. It is straightforward that the space $Y = ϕ(X)$ is scattered and has the dispersion index $≤ n$ (it is not so straightforward that $Y$ is Eberlein compact but this is known) so the set $L' = ϕ(L)$ is $σ$-compact by the induction hypothesis. Choose a family $\{K_n : n \in \omega\}$ of compact subsets of $Y$ such that $L' = \bigcup_{i \in \omega} K_i$ and let $M_i = (K_i \setminus \{q\}) \cup X^{(n-1)}$; it follows from $M_i = ϕ^{-1}(K_i)$ that $M_i$ is compact for every $i \in \omega$. Therefore $L_i = (K_i \setminus \{q\}) \cup Q = L \cap M_i$ so $L_i$ is Lindelöf for any $i \in \omega$.

The space $X$ must be hereditarily metacompact (see [13, Theorem 8]) so we can find a clopen set $U_x$ such that $U_x \cap X^{(n-1)} = \{x\}$ for any $x \in X^{(n-1)} \setminus Q$ and the family $\mathcal{U} = \{U_x : x \in X^{(n-1)} \setminus Q\}$ is point-finite. It follows from the Lindelöf property of $L_i$ that every $x \in X^{(n-1)} \setminus Q$ can be separated from $L_i$ by a $G_δ$-set so we can find a decreasing family $\{V_j^x : j \in \omega\}$ of clopen neighborhoods of $x$ such that $V_0^x \subseteq U_x$ and $\bigcap_{j \in \omega} V_j^x \cap L_i = \emptyset$.

Consider the set $W_j = \bigcup\{V_j^y : y \in X^{(n-1)} \setminus Q\}$; given any point $y \in L_i$, there is a finite set $B \subseteq X^{(n-1)} \setminus Q$ such that $y \notin U_x$ for any $x \in X^{(n-1)} \setminus (Q \cup B)$ and hence we can find $j \in \omega$ such that $y \notin V_j^x$ for any $x \in B$. This implies that $y \notin W_j$ so $\bigcap_{k \in \omega} W_k \cap L_i = \emptyset$; since $X^{(n-1)} \setminus Q \subset W = \bigcap_{k \in \omega} W_k$, we have the equality $L_i = M_i \setminus W$ and hence $L_i$ is $σ$-compact being an $F_σ$-subset of a compact space $M_i$. Therefore $L = \bigcup_{n\in\omega} L_n$ is $σ$-compact as required. \hfill $\Box$

3.8. **Lemma.** Given spaces $X$ and $Y$, if $f : X \to Y$ is a continuous map then $f : (X)_κ \to (Y)_κ$ is also continuous.

**Proof.** The family $\mathcal{B}$ of all $G_κ$-subsets of $Y$ is a base in $(Y)_κ$. If $B \in \mathcal{B}$ then $f^{-1}(B)$ is a $G_κ$-subset of $X$ so $f^{-1}(B)$ is open in $(X)_κ$. \hfill $\Box$

3.9. **Theorem.** If $K$ is a scattered Eberlein compact space then its $ω$-modification is a monotonically Sokolov space.

**Proof.** Denote by $X$ the $ω$-modification of $K$; then $X$ is a Lindelöf $P$-space (see [12, Proposition 1]). If $F$ is closed in $X$ then it is Lindelöf and hence $F$ is also Lindelöf considered as a subspace of $K$. By Theorem 3.7 we can find a countable family $\mathcal{C}_F$ of compact subspaces of $K$ such that $F = \bigcup \mathcal{C}_F$. The space $K$ is $σ$-discrete by Proposition 8 of [6] so we can fix for any set $A \subseteq X$ a countable family $\mathcal{D}(A)$ of discrete subsets of $X$ such that $A = \bigcup \mathcal{D}(A)$.

The space $K$ being monotonically Sokolov by Theorem 5.4 of [5], we can assign to any family $\mathcal{F}$ of closed subspaces of $K$ a retraction $r_\mathcal{F} : K \to K$ and
Observe that every retraction \( r_F \) in \( X \) satisfies the Corollary.

3.10. \textbf{Corollary.} For any scattered Eberlein compact space \( K \), if we denote by \( X \) the \( \omega \)-modification of \( K \), then \( C_p(X) \) is normal and hence \( \text{ext}(C_p(X)) = \omega \).

\textit{Proof.} The space \( X \) is monotonically Sokolov by Theorem 3.9 so it is Sokolov by Corollary 4.20 of [5]. Therefore \( C_p(X) \) is also Sokolov by [10, Theorem 2.1(d)] and hence it is normal and has countable extent by [10, Proposition 2.2]. \( \square \)

3.11. \textbf{Observation.} Theorem 3.7 exhibits a very strong restriction on Lindelöf subspaces of a scattered Eberlein compact space. Observe first that if we omit “scattered” from the assumption then Theorem 3.7 is no longer true because any second countable space embeds in an Eberlein compact space. We can, however, conjecture that every Lindelöf subspace of an Eberlein compact space is Lindelöf \( \Sigma \). It is not clear whether this is true even for Corson compact spaces.

Recall that a compact space \( X \) is Eberlein compact if and only if it embeds in \( C_p(K) \) for some compact space \( K \). It is worth noting that we cannot expect that Lindelöf property of \( L \) implies that \( L \) is a Lindelöf \( \Sigma \)-space if \( L \) is embeddable in \( C_p(K) \) for some compact space \( K \) because even \( C_p(K) \) can be Lindelöf without being Lindelöf \( \Sigma \) (see [2, Example IV.7.1]).

We will next give an example of a Lindelöf \( P \)-space which fails to be Sokolov. Our main tool is the following simple characterization of closed discrete subsets of function spaces.
3.12. Proposition. If $X$ is a Lindelöf $P$-space then an infinite set $D \subset C_p(X)$ is closed and discrete if and only if for any infinite $E \subset D$ there is a point $x \in X$ such that the set $\{ f(x) : f \in E \}$ is not bounded in $\mathbb{R}$.

Proof. Assume first that $D$ is closed and discrete in $C_p(X)$ while there is a countably infinite $E \subset D$ such that the set $E_x = \{ f(x) : f \in E \}$ is bounded in $\mathbb{R}$ and hence the set $K_x = E_x$ is compact for every point $x \in X$. It follows from the $P$-property of $X$ that the closure $Q$ of the set $E$ is $\mathbb{R}^X$ is contained in $C_p(X)$. It is immediate that $Q$ is contained in the compact set $\prod \{ K_x : x \in X \}$ and hence $Q$ is compact so $E$ has a cluster point in $Q$ and hence in $C_p(X)$ which is a contradiction.

To prove sufficiency, suppose that $D$ has a cluster point $h$ in $C_p(X)$; since the space $C_p(X)$ is Fréchet–Urysohn (see [2, Theorem II.7.15]), there is a non-trivial sequence $E \subset D$ which converges to $h$. Therefore $E$ is an infinite set and it follows from compactness of $E \cup \{ h \}$ that the set $\{ f(x) : f \in E \cup \{ h \} \}$ is bounded in $\mathbb{R}$ for any point $x \in X$. Therefore every set $E_x$ is bounded in $\mathbb{R}$ as promised. \qed

3.13. Example. Denote by $C$ the set of all countable limit ordinals; for any $\lambda \in C$ consider an increasing sequence $S_\lambda = \{ \mu_\alpha(\lambda) : n \in \omega \}$ of successor ordinals converging to $\lambda$ and denote by $P$ the set $\{ x \in \mathbb{D}^\omega : \vert x^{-1}(1) \vert < \omega \}$. Let $f_\lambda(\alpha) = 1$ if $\alpha \in S_\lambda$ and $f_\lambda(\alpha) = 0$ whenever $\alpha \in \omega_1 \setminus S_\lambda$, i.e., $f_\lambda \in \mathbb{D}^\omega$ is the characteristic function of the set $S_\lambda$. Let $X$ be the set $P \cup \{ f_\lambda : \lambda \in C \}$ with the $\omega$-modification of the topology of subspace of $\mathbb{D}^\omega$. It is evident that $X$ is a $P$-space; Telgarsky proved (see [9, Theorem 7.1.]) that $X$ is Lindelöf. It turns out that $\text{ext}(C_p(X)) > \omega$ and hence neither $X$ nor $C_p(X)$ is a Sokolov space.

Proof. For any $x \in X$ and ordinal $\alpha < \omega_1$ the set $O_\alpha(x) = \{ y \in X : y(\beta) = x(\beta) \}$ for any $\beta < \alpha$ is an open neighborhood of the point $x$ in $X$ and the family $\{ O_\alpha(x) : \alpha < \omega_1 \}$ is a local base of $X$ at $x$. Given an ordinal $\lambda \in C$ and a finite set $K \subset \lambda$ denote by $\chi_K$ the characteristic function of $K$; it is immediate that the set $O_\lambda(\chi_K)$ is clopen in $X$. Therefore $U(\lambda, n) = \bigcup \{ O_\lambda(\chi_K) : K \text{ is a subset of } \lambda \text{ of cardinality } n \}$ is also clopen for any $n \in \omega$ being the countable union of clopen sets. Observe that $U(\lambda, n) \cap U(\lambda, m) = \emptyset$ for distinct $m$ and $n$ and $G(\lambda) = \bigcup \{ U(\lambda, n) : n \in \omega \}$ covers the set $P \cup \{ f_\mu : \mu > \lambda \}$.

Given $n \in \omega$ let $\xi_\lambda(x) = n$ if $x \in U(\lambda, n)$ and let $\xi_\lambda(x) = 0$ for any point $x \in \{ f_\mu : \mu \leq \lambda \}$. Since the points of the set $\{ f_\mu : \mu < \omega_1 \}$ are isolated in $X$, the function $\xi_\lambda : X \to \mathbb{R}$ is continuous for any $\lambda \in C$. If $\lambda < \gamma$ then take an ordinal $\alpha \in [\lambda, \gamma)$ and let $K = \{ \alpha \}$. Then $\xi_\lambda(\chi_K) = 0$ and $\xi_\gamma(\chi_K) = 1$ which shows that $\xi_\lambda \neq \xi_\gamma$ so the set $D = \{ \xi_\lambda : \lambda \in C \} \subset C_p(X)$ is uncountable. We claim that $D$ is closed and discrete.

Indeed, if $A$ is an infinite subset of $C$, then we can choose an increasing sequence $\{ \lambda_n : n \in \omega \} \subset A$. If $\lambda = \sup \{ \lambda_n : n \in \omega \}$, then $\lambda \in C$ and the
set $K_n = S_{\lambda} \cap \lambda_n$ is finite; if $r_n = |K_n|$ then $\xi_{\lambda_n}(f_\lambda) = r_n$ for any $n \in \omega$. It follows from the equality $S_{\lambda} = \bigcup\{K_n : n \in \omega\}$ that the sequence $\{r_n : n \in \omega\}$ is unbounded so we can apply Proposition 3.12 to conclude that $D$ is closed and discrete. \qed

The following corollary answers Problem 3.12 from [10].

3.14. Corollary. There exists a Lindelöf $P$-space which does not have the Sokolov property.

Proof. Let $X$ be the Lindelöf $P$-space from Example 3.13; then $\text{ext}(C_p(X)) > \omega$. If $X$ is Sokolov then $C_p(X)$ is also Sokolov (see [10, Theorem 2.1(d)]) and hence $\text{ext}(C_p(X)) \leq \omega$ (see Proposition 2.2 of [10]) which is a contradiction. \qed

The example below gives a negative answer to Problem 3.13 from [10].

3.15. Example. Denote by $C$ the set of all countable limit ordinals and suppose that $\Omega \subset C$. For any $\lambda \in \Omega$ pick an increasing sequence $S_{\lambda} = \{\mu_n(\lambda) : n \in \omega\}$ of successor ordinals converging to $\lambda$. Define a topology $\tau$ on $\omega_1$ by declaring all points of $\omega_1 \setminus \Omega$ isolated and for any ordinal $\lambda \in \Omega$ let the local base at $\lambda$ be the family $\{\{\lambda\} \cup (S_{\lambda} \setminus F) : F$ is a finite subset of $S_{\lambda}\}$. It is immediate that $X = (\omega_1, \tau)$ is a locally compact space; denote by $K(\Omega)$ its one-point compactification. Sokolov proved (see Proposition 4.2 of [8]) that there exists a set $\Omega \subset \omega_1$ such that the space $K(\Omega)$ fails to be Sokolov. However, it is a result of Leiderman (see [4, Example 3]) that $K(\Omega)$ embeds in $C_p(L)$ for some Lindelöf $P$-space $L$. Therefore a compact subspace of $C_p(L)$ can fail to be Sokolov for a Lindelöf $P$-space $L$.

4. Open Problems

The author hopes that this paper can convince the reader that it is an interesting task to find out when the Sokolov property holds in a Lindelöf $P$-space. The following list of open problems shows that we are still far away from having a complete picture of the topic.

4.1. Question. Suppose that $K$ is a scattered compact space and let $X$ be the $\omega$-modification of $K$. Must the space $X$ be monotonically Sokolov?

4.2. Question. Suppose that $K$ is a scattered compact space and let $X$ be the $\omega$-modification of $K$. Must the space $X$ be Sokolov?

4.3. Question. Suppose that $K$ is a scattered compact space and let $X$ be the $\omega$-modification of $K$. Is it true that $\text{ext}(C_p(X)) = \omega$?

4.4. Question. Suppose that $K$ is a scattered compact space of finite dispersion index and let $X$ be the $\omega$-modification of $K$. Must the space $X$ be monotonically Sokolov?
4.5. **Question.** Suppose that $K$ is a scattered compact space of finite dispersion index and let $X$ be the $\omega$-modification of $K$. Must the space $X$ be Sokolov?

4.6. **Question.** Suppose that $K$ is a scattered compact space of finite dispersion index and let $X$ be the $\omega$-modification of $K$. Is it true that $\text{ext}(C_p(X)) = \omega$?

4.7. **Question.** Assume that $\alpha$ is an ordinal and $K = \alpha + 1$ is considered with the topology generated by the natural order on $\alpha + 1$. Then $K$ is a scattered compact space; let $X$ be the $\omega$-modification of $K$. Must the space $X$ be monotonically Sokolov?

4.8. **Question.** Suppose that $K$ is a Corson compact space and $L \subset K$ is Lindelöf. Must $L$ be a Lindelöf $\Sigma$-space?

4.9. **Question.** Suppose that $K$ is a Corson compact space and $L \subset K$ is Lindelöf. Must $L^n$ be Lindelöf for any $n \in \mathbb{N}$?

4.10. **Question.** Suppose that $K$ is an Eberlein compact space and $L \subset K$ is Lindelöf. Must $L$ be a Lindelöf $\Sigma$-space?

**References**

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