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A HISTORICAL PERSPECTIVE ON THE PROBLEM OF REPRESENTING THE ROOTS OF UNITY THROUGH RADICALS

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Abstract. The aim of this work is to provide a concise perspective on some historical developments toward the determination of those angles θ for which $\sin\theta$ (and thus, also $\cos\theta$) may be expressed in terms of radicals. This is clearly a particular task in the investigation of conditions under which the roots of the polynomial $x^n = 1$ may be expressed through radicals, for $n \in \mathbb{Z}^+$. Departing from the general formula for the roots of quadratic polynomials with real coefficients, we evoke important efforts related to the solution of this problem, like the problem of ruler and compass constructibility as well as individual approaches due to De Moivre, Vandermonde and Gauss. We close the present note providing an easy, partial, affirmative solution to the problem in question.

1. Introduction

The emergence of new problems in scientific areas in general and, in particular, in the mathematical sciences, is an every-day event. In fact, any biographical investigation on the research achievements of famous mathematicians, shows that the satisfactory resolution of a particular problem gives rise to many questions on the problem itself (limitations, possible directions of generalization, determination of examples and counter-examples, etc.) and on problems of parallel interest derived during the course of the investigation. Examples of these facts are abundant in the history of mathematics and science, and they evince that the mathematical and scientific tasks yet to be accomplished are still great in number. In most of the cases, the problems possess a high complexity which requires the use of sophisticated mathematical tools for both its resolution and its introduction. For those problems, the historical background is hardly relevant; in fact, what matters is the set of recent results that lead to prove the new results reported. On the other hand, problems like Fermat's last theorem are also difficult to establish, but easy to present to a general audience [16].

In this work, we state a mathematical problem which is easy to describe to a general audience of undergraduate students. The problem, in many respects, resembles the traditional exercises of Galois theory on the radical representation of roots of polynomials, but the authors have not been able to find its general solution in the context of Galois groups. Looking for an affirmative answer to the problem of interest, the authors devoted part of their efforts to investigate several traditional approaches to the task. In this way, we became involved (and interested) in the historical evolution of the problem of representing certain roots of unity with radicals. The present report is thus a summary on some historical developments that converge to understanding the problem and the possible methods of solution. Here, we will state individual efforts by several mathematicians which have contributed significantly to the theory of equations, and will try to put together their approaches to provide a concise perspective on the solution of the problem.

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Palabras clave. Historical perspective of a mathematical problem, development in pure and applied mathematics, solubility of equations by radicals, roots of unity, constructibility by ruler and compass.

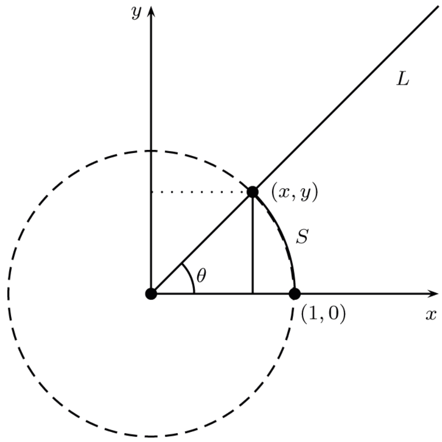

FIGURE 1. Definition of the trigonometric functions.

We give a fresh start to this discussion by introducing the problem. Consider the unit circle and an angle θ formed with the x-axis, and a segment of line L obtained by a rotation of the x-axis counterclockwise. Let (x, y) be the point of intersection of the unit circle and the segment L. We say that θ is equal to S radians, where S is the arc-length between the points (x, y) and $(1, 0)$ (see Figure 1). The trigonometric functions of θ are defined as

$$\sin\theta = y, \qquad \cos\theta = x.$$

These definitions extend the Greek's definitions of the trigonometric functions to angles of any measure. Some particular values may be obtained using this definition and some trigonometric identities. For example, using the identity $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$ with $\theta = \pi/3$ and $x = \sin(\pi/3)$, one may readily verify that $0 = 3x - 4x^3 = x(3-4x^2)$. The solutions to this equation are $0, -\sqrt{3}/2, \sqrt{3}/2$, but the only admissible (positive) solution is $\sqrt{3}/2$; therefore $\sin(\pi/3) = \sqrt{3}/2$. Using the same idea, we can check that

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \qquad \sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}, \qquad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

A natural question arises in this context: *For which angles θ are the trigonometric functions $\sin\theta$ and $\cos\theta$ expressible in terms of radicals?*

This paper is organized as follows. In Section 2, we provide a brief historical review on the development of methods of solutions of polynomial functions. Starting from the general formula for quadratic equations, we recall successful efforts by Scipone del Ferro, Niccolò Fontana of Brescia (Tartaglia), Gerolamo Cardano and Ludovico Ferrari to represent the roots of polynomials by radicals involving the coefficients of the polynomial. The connections between the roots of unity and ruler-and-compass constructions are examined in Section 3. We take a look therein at the de Moivre's formula to determine the nth roots of 1 and, in a parallel stage, at the constructibility of the regular pentagon. Vandermonde's method is introduced in Section 4, together with an application to the radical representation of the 11th roots of unity. Following our historical perspective, Section 5 presents Gauss' method of radical representation of the roots of unity. Finally, we close this manuscript with a section of concluding remarks which converge to the problem that has motivated this historical discussion.

2. Unsolvability of the quintic

Physical evidence suggests that, as early as 2000 BC, Babylonian mathematicians were able to solve some second-degree equations arising from daily-life tasks [6].

However, humankind had to wait until the 12th century A. D. to possess a means to solve the general, quadratic equation

$$ax^2 + bx + c = 0.$$

Based largely on previous efforts by Al-Khwarizmi (c. 780–c. 850) [13], the Jewish mathematician Abraham bar Hiyya Ha-Nasi (c. 1065–1136) provided a sequence of instructions to determine the solutions of the general quadratic equation. The details were published in his manuscript *Treatise on measurement and calculation*, which is considered the first European recording of the resolution of general, quadratic polynomials [14]. Nowadays, we know that the solutions are given by the formula

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The beginning of the formal investigation of equations still had to wait several centuries. In part, this was due to the lack of an operational nomenclature, and the limitations of the mathematical language to state and solve problems. The mathematical development of the theory and its notation was hard and slow, and its history is carved with small individual efforts that resulted in the modern theory of equations [9, 21]. For instance, the symbol = was introduced in 1557 by the English mathematician Robert Recorde (c. 1512–1558) [12], and François Viète (1540–1603) introduced the concept of literal constants in equations [3] in 1591.

It is historically important to mention that Scipione del Ferro (1465–1526) [10] is believed to have solved third-degree equations of the form

$$x^3 + px = q, \tag{1}$$

for $p, q \in \mathbb{R}^+$. However, del Ferro never published his investigations but before his death he communicated the results of some of his studies to his student Antonio Maria Fiore [5]. In 1535, Fiore challenged Niccolò Fontana of Brescia (c. 1500–1557), also known as Tartaglia, to a public contest in which each of the two men had to solve 30 mathematical problems proposed by the other. Tartaglia, a mathematics teacher in Venice, had apparently discovered the solution to cubic equations of the form

$$x^3 + bx^2 = q,$$

with $b, q \in \mathbb{R}^+$, and Fiore, keen to obtain a good teaching position in his native Venice, was confident to reach his goal by publicly humiliating his opponent.

The night before the contest, Tartaglia discovered a method to solve (1). Being all problems of that form, it took Tartaglia less than two hours to solve all the problems proposed by Fiore. The equations proposed by Tartaglia, on the other hand, were more diverse in form and difficulty, and Fiore could not solve the entire list. Tartaglia's method of solution of cubic polynomials was popularized in 1545 by Gerolamo Cardano (1501–1576) in his work *Ars Magna*, a landmark in the history of algebra [18] that also reports on Ludovico Ferrari's (1522–1565) method to solve fourth-degree equations. Indeed, Cardano's work is considered the formal beginning in the investigation of equations.

After these successful efforts in the resolution of cubic and quartic polynomials, many mathematicians tried to find a general formula for the quintic equation [20]. Using Joseph Louis Lagrange's (1736–1813) approach on permutations, Paolo Ruffini (1765–1822) almost demonstrated the insolubility by radicals of the general polynomial equations of degree five or higher [2]. His proofs contained several mistakes which were fixed later in 1823 by Niels Henrik Abel (1802–1829) [23]. Finally, Évariste Galois (1811–1832) developed a general method to determine when an arbitrary polynomial equation has solutions by radicals [22].

3. Ruler-and-compass constructions

The representation of the solutions of equations by means of radicals is an interesting problem related to the problem of constructibility of geometric figures by ruler and compass. For the sake of completeness, we devote the present section to outline the relevant relationships between these two problems, since an in-depth treatise is not part of the aims of this manuscript (see [25] for a formal study).

The ancient Greek used only an ungraded ruler and a compass to construct geometric figures. The rules for performing these constructions are the following [1]:

- Given two previously constructed points, one can construct the line segment joining them; evidently, if this segment intersects a line segment previously constructed, then one readily constructs their point of intersection.
- Given a previously constructed segment and a given point, we can construct a circle with center at the point and radius equal to the length of the segment. Moreover, if the circle intersects a line segment or a circle previously constructed, then their intersection points are thus constructed.
- We can construct new points by intersecting a previously constructed line or circle, with a previously constructed segment extended in both directions.
- These rules of constructibility can only be applied a finite number of times.

Example 1. The following are examples of ruler-and-compass constructions [26]:

(a) A line perpendicular to a previously constructed line.
(b) The bisection of an angle.
(c) Finding the midpoint of a segment. □

We say a *length is constructible* if it can be obtained from a finite number of applications of the ruler-and-compass rules. We say that a *number is constructible* if a segment of such length is constructible. Clearly, any positive integer number n is constructible, its length being obtained through the juxtaposition of n segments of unit length. Example 1(c) shows that $\frac{1}{2}$ in particular and, in general, any rational number can also be built with a ruler and a compass. Moreover, if $a \in \mathbb{R}^+$ can be constructed, then so can $\sqrt{a}$, as shown by Figure 2.

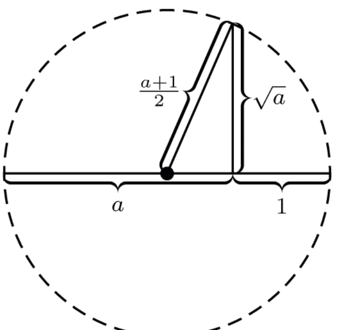

FIGURE 2. Ruler-and-compass construction of $\sqrt{a}$.

Remark 2. The number $\sqrt[3]{2}-1$ is not constructible using ruler and compass. Thus, for an angle θ equal to $\cos^{-1}(\sqrt[3]{2}-1)$, the number $\cos\theta$ cannot be expressed in terms of integer numbers, the four elementary operations and square roots, exclusively. In fact, the presence of cubic roots is necessary to provide such representation. We are interested in deciding whether the number $\cos(\sqrt[3]{2}\pi)$ is expressible by means of radicals. □

3.1. **De Moivre's method.** Abraham de Moivre (1667–1754) was one of the first mathematicians to investigate the roots of the equation $x^n = 1$, for $n \in \mathbb{Z}^+$. These special roots are called the *nth roots of* 1, and they are given by the well-known de Moivre's formula [11]. Among many other mathematical achievements [24], de Moivre proved that the roots of the equation

$$x^5 = 1 \tag{2}$$

can be expressed by radicals.

Since $x_1 = 1$ is a solution of (2) then $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1) = 0$. In this way, the expression to be solved is $x^4 + x^3 + x^2 + x + 1 = 0$. Dividing by x^2, we obtain

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1 = x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0.$$

Using the change of variable $y = x + 1/x$, one readily reaches the equation $y^2 + y - 1 = 0$, whose solutions are the numbers $y_1 = (\sqrt{5} - 1)/2$ and $y_2 = -(\sqrt{5} + 1)/2$. In terms of the original variable x, one obtains the following equations:

$$x^2 - y_1 x + 1 = 0, \qquad x^2 - y_2 x + 1 = 0.$$

Using again the general formula for second-degree equations, we obtain the other four roots:

$$\begin{aligned} x_2 &= \frac{1}{4}(\sqrt{5} - 1) - \frac{1}{4} i \sqrt{\sqrt{20} + 10}, & x_3 &= -\frac{1}{4}(\sqrt{5} + 1) - \frac{1}{4} i \sqrt{10 - \sqrt{20}}, \\ x_4 &= -\frac{1}{4}(\sqrt{5} + 1) + \frac{1}{4} i \sqrt{10 - \sqrt{20}}, & x_5 &= \frac{1}{4}(\sqrt{5} - 1) + \frac{1}{4} i \sqrt{\sqrt{20} + 10}. \end{aligned}$$

This discussion clearly illustrates the existence of quintic equations whose solutions can be expressed using radicals.

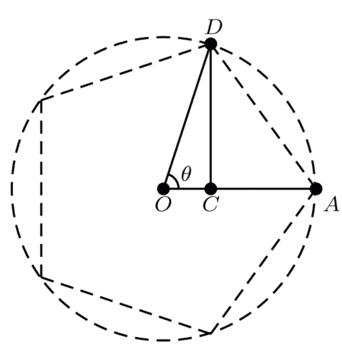

FIGURE 3. Ruler-and-compass construction of the pentagon. In this figure, $\theta = 2\pi/5$ and the segment OC has length equal to $\cos(2\pi/5)$.

The case of the equation $x^6 = 1$ is trivial. Indeed, notice that we may rewrite it as $(x^2)^3 = 1$. If we let $y = x^2$, then $y^3 = 1$. As before, since 1 is a solution for this last equation, then the problem reduces to solving $y^2 + y + 1 = 0$. Using the quadratic formula, one readily obtains that $y_1 = -(1 + i\sqrt{3})/2$ and $y_1 = -(1 - i\sqrt{3})/2$, whence the following equations result:

$$x^2 = 1, \qquad x^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \qquad x^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

The six solutions to these equations are

$$\begin{aligned} x_1 &= -1, & x_2 &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}, & x_3 &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \\ x_4 &= 1, & x_5 &= \frac{1}{2} - i\frac{\sqrt{3}}{2}, & x_6 &= \frac{1}{2} + i\frac{\sqrt{3}}{2}. \end{aligned}$$

Remark 3. Using the same method to solve (2), de Moivre proved that the seventh roots of unity can also be expressed by radicals [26], though cubic roots of complex numbers appear in this case. □

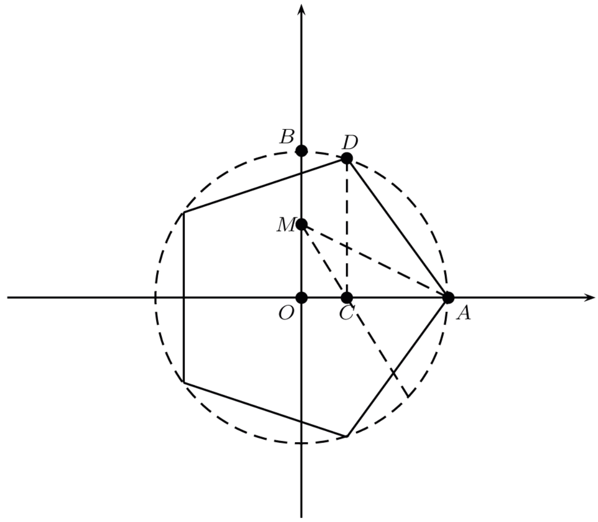

FIGURE 4. A method for the ruler-and-compass construction of the pentagon.

3.2. **Construction of the regular pentagon.** The ruler-and-compass construction of a regular pentagon is carried out in Figure 3, where $\theta = 2\pi/5$. One readily sees that the pentagon is constructible if the segment OC (which has length equal to $\cos(2\pi/5)$) can be constructed. In fact, it suffices to trace a line perpendicular to the segment OA that passes through the point C, in order to determine D (whose x- and y-coordinates are $\cos(2\pi/5)$ and $\sin(2\pi/5)$, respectively). In this way, the segment AD becomes one of the sides of the regular pentagon. On the other hand, de Moivre's formula guarantees that $\cos(2\pi/5) + i\sin(2\pi/5)$ is a root of Equation (2), which means that either $\cos\theta$ is equal to $\frac{1}{4}(\sqrt{5}-1)$, or $-\frac{1}{4}(\sqrt{5}+1)$. Consequently, $\cos\theta = \frac{1}{4}(\sqrt{5}-1)$. Since 5 is constructible then so is $\sqrt{5}$; thus, $\cos\theta$ can be constructed using ruler and compass. In fact, the construction can be performed following the rules of Example 1 (see Figure 4):

I Consider a unit circle with center O.
II Draw a segment of line OA, and a perpendicular segment OB of the same length of the segment OA, that passes through O.
III Find the midpoint M of the segment OB, and trace the segment AM.
IV Bisect the angle OMA, and let C be the point of intersection with the segment OA.
V Trace a segment perpendicular to OA that passes through C, and let D be the point of intersection with the unit circle. In this way, D and A will be two vertices of the pentagon.

Before we close this section, it is worth mentioning that the regular heptagon cannot be constructed by ruler and compass. This follows from the fact that the number $2\cos(2\pi/7)$ is a zero of the irreducible cubic polynomial $x^3 + x^2 - 2x - 1$.

4. VANDERMONDE'S METHOD

In the present section, we will show that the method developed by Alexandre-Théophile Vandermonde (1735–1796) [8] can be adapted to establish that the roots of the equation $x^{11} = 1$ may be represented through radicals. Our approach will hinge on the use of Lagrange's resolvent.

One of the first successful efforts to relate the coefficients of a polynomial to its roots was achieved by François Viète. In general, he considered a polynomial of degree n over $\mathbb{R}$ of the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n. \tag{3}$$

By the Fundamental Theorem of Algebra, this polynomial has n (not necessarily different) complex roots which may be denoted by $x_1, x_2, \ldots, x_n$. Viète established that the coefficient a_{n-k} satisfies the formula

$$\sum_{1\le m_1<m_2<\ldots<m_k\le n} \prod_{i=1}^{k} x_{m_i} = (-1)^k \frac{a_{n-k}}{a_n},$$

for each $k = 1, 2, \ldots, n$. However, Lagrange's contributions to the investigation of radical representations of roots of polynomials were more interesting and innovative. Indeed, while searching for a general formula to find the roots of polynomials of degree n, Lagrange was one of the first mathematicians to notice the importance of the symmetries of the roots of polynomials. In our approach, we take a fresh start by considering a general polynomial of the form (3).

Let $\eta_1, ..., \eta_n$ be the roots of the polynomial $p(x)$. The Lagrange resolvent [15] is defined by

$$t(w) = \eta_1 + w\eta_2 + w^2\eta_3 + \cdots + w^{n-1}\eta_n,$$

where w is an nth root of 1. It is not difficult to verify that

$$\eta_j = \frac{1}{n}\sum_{w} w^{-(j-1)} t(w), \tag{4}$$

where the summation runs over all the roots of unity. As we noticed before, it suffices to study the roots of the cyclotomic polynomial

$$x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0. \tag{5}$$

Dividing this polynomial by x^4 and using the change of variable $y = x + 1/x$, we obtain the quintic equation

$$y^5 + y^4 - 4y^3 - 3y^2 + 3y + 1 = 0. \tag{6}$$

For each $\theta \in \mathbb{R}$ define $e^{i\theta} = \cos\theta + i\sin\theta$. Since the roots of Equation (5) are of the form $e^{2k\pi i/11}$ for $k = 1, \ldots, 11$, then the roots of (6) take the form

$$\begin{aligned} e^{2k\pi i/11} + \frac{1}{e^{2k\pi i/11}} &= \cos\left(\frac{2k\pi}{11}\right) + i\sin\left(\frac{2k\pi}{11}\right) + \cos\left(-\frac{2k\pi}{11}\right) + i\sin\left(-\frac{2k\pi}{11}\right) \\ &= 2\cos\left(\frac{2k\pi}{11}\right). \end{aligned}$$

Let $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ be the roots of (6). By letting

$$\eta_1 = 2\cos\left(\frac{2\pi}{11}\right), \quad \eta_2 = 2\cos\left(\frac{4\pi}{11}\right),$$

$$\eta_3 = 2\cos\left(\frac{6\pi}{11}\right), \quad \eta_4 = 2\cos\left(\frac{8\pi}{11}\right), \quad \eta_5 = 2\cos\left(\frac{10\pi}{11}\right),$$

and using the formula $2\cos\theta\cos\varphi = \cos(\theta+\varphi) + \cos(\theta-\varphi)$, Vandermonde proved the following relations between the roots:

$$\begin{cases} \eta_1^2 = \eta_2 + 2, & \eta_2^2 = \eta_4 + 2, & \eta_3^2 = \eta_5 + 2, \\ \eta_1\eta_2 = \eta_1 + \eta_3, & \eta_1\eta_3 = \eta_2 + \eta_4, & \eta_1\eta_4 = \eta_3 + \eta_5, \\ \eta_2\eta_3 = \eta_1 + \eta_5, & \eta_2\eta_4 = \eta_2 + \eta_5, & \eta_2\eta_5 = \eta_3 + \eta_4, \\ \eta_3\eta_4 = \eta_1 + \eta_4, & \eta_3\eta_5 = \eta_2 + \eta_3, & \eta_4^2 = \eta_3 + 2, \\ \eta_4\eta_5 = \eta_1 + \eta_2, & \eta_5^2 = \eta_1 + 2, & \eta_1\eta_5 = \eta_4 + \eta_5. \end{cases} \tag{7}$$

Surprisingly enough, Vandermonde discovered that the permutation

$$\eta_1 \mapsto \eta_2 \mapsto \eta_4 \mapsto \eta_3 \mapsto \eta_5 \mapsto \eta_1 \tag{8}$$

of the roots, preserves the relations between them. For example, if we apply this permutation to the expression $\eta_1\eta_4 = \eta_3 + \eta_5$, then the resulting relation $\eta_2\eta_3 = \eta_5 + \eta_1$ is also a valid identity, according to the list (7).

The Lagrange resolvent of (6) is given by

$$V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5) := t(w^k) = \eta_1 + w^k\eta_2 + w^{2k}\eta_3 + w^{3k}\eta_4 + w^{4k}\eta_5,$$

for $k = 1,\ldots,5$ [2], with $w = e^{2\pi i/5}$. The relations (7) imply that

$$\begin{aligned} V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5 &= \left(\eta_1 + w^k\eta_2 + w^{2k}\eta_3 + w^{3k}\eta_4 + w^{4k}\eta_5\right)^5 \\ &= C_1\eta_1 + C_2\eta_2 + C_3\eta_3 + C_4\eta_4 + C_5\eta_5 + C_6, \end{aligned}$$

where the coefficients C_i are polynomial functions in w. Four applications of the permutation (8) yield

$$\begin{aligned} V_k(\eta_2,\eta_4,\eta_5,\eta_3,\eta_1)^5 &= C_1\eta_2 + C_2\eta_4 + C_3\eta_5 + C_4\eta_3 + C_5\eta_1 + C_6, \\ V_k(\eta_4,\eta_3,\eta_1,\eta_5,\eta_2)^5 &= C_1\eta_4 + C_2\eta_3 + C_3\eta_1 + C_4\eta_4 + C_5\eta_2 + C_6, \\ V_k(\eta_3,\eta_5,\eta_2,\eta_1,\eta_4)^5 &= C_1\eta_3 + C_2\eta_5 + C_3\eta_2 + C_4\eta_1 + C_5\eta_4 + C_6, \\ V_k(\eta_5,\eta_1,\eta_4,\eta_2,\eta_3)^5 &= C_1\eta_5 + C_2\eta_1 + C_3\eta_4 + C_4\eta_2 + C_5\eta_3 + C_6. \end{aligned} \tag{9}$$

On the other hand, an application of the permutation (8) to $V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)$ results in the identities

$$\begin{aligned} V_k(\eta_2,\eta_4,\eta_5,\eta_3,\eta_1) &= \eta_2 + w^k\eta_4 + w^{2k}\eta_3 + w^{3k}\eta_5 + w^{4k}\eta_1 \\ &= w^{-1}\left(\eta_2 w + w^{2k}\eta_4 + w^{3k}\eta_3 + w^{4k}\eta_5 + w^{5k}\eta_1\right) \\ &= w^{-1}V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5). \end{aligned}$$

As a consequence,

$$\begin{aligned} V_k(\eta_2,\eta_4,\eta_5,\eta_3,\eta_1)^5 &= w^{-5}V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5 \\ &= V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5, \end{aligned}$$

that is, the permutation (8) leaves $V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5$ invariant. This and Equations (9) establish that

$$5V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5 = (C_1 + C_2 + C_3 + C_4 + C_5)(\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5) + 5C_6.$$

But the coefficient of y^4 in (6) is equal to 1, so $\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 = 1$ [27]. Thus,

$$V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5 = \frac{1}{5}(C_1 + C_2 + C_3 + C_4 + C_5) + C_6.$$

As a consequence, $V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5$ is expressed in terms of a polynomial that depends on ω, which is a 5th root of 1. Therefore, an application of de Movrie's formula yields that $V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5$ can also be expressed in terms of radicals. Finally, applying Lagrange's formula (4), we obtain that

$$\eta_j = \frac{1}{5}\sum_{k=1}^{5}\omega^{-(j-1)k}\sqrt[5]{V_k(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^5},$$

for every $j = 1,\ldots,5$. We conclude that all the 11th roots of 1 can also be expressed using radicals.

5. Gauss' method

Carl Friedrich Gauss (1777-1855) established many important contributions to the theory of equations. His idea of congruence, for instance, has been used in the theory of numbers, and it has been extended to other algebraic scenarios [19]. Recall that two integer numbers a and b are congruent modulo an integer $n \neq 0$ if $a - b$ is divisible by n. Gauss denoted this relation by $a \equiv b \bmod n$.

Another important concept introduced by Gauss was that of a primitive root. An integer number g is called a *primitive root modulo* a prime number p if $g^{p-1} \equiv 1 \bmod p$ and $g^i \not\equiv 1 \bmod p$, for every $i = 1,\ldots,p-2$. For every prime number p there is a primitive root g modulo p, and $g^0, g^1, g^2, \ldots, g^{p-2}$ are congruent with $1, 2, 3,\ldots, p-1$ modulo p (the congruences are not necessarily in order). This induces the function $\vartheta : \{1,2,\ldots,p-1\} \to \{0,1,2,\ldots,p-2\}$, given by

$$g^{\vartheta(k)} \equiv k \bmod p. \tag{10}$$

Example 4. If $p = 11$ then 2 is a primitive root and, in this case,

$$\begin{array}{ll} 2^0 \equiv 1 \bmod 11, & 2^9 \equiv 6 \bmod 11, \\ 2^1 \equiv 2 \bmod 11, & 2^7 \equiv 7 \bmod 11, \\ 2^8 \equiv 3 \bmod 11, & 2^3 \equiv 8 \bmod 11, \\ 2^2 \equiv 4 \bmod 11, & 2^6 \equiv 9 \bmod 11, \\ 2^4 \equiv 5 \bmod 11, & 2^5 \equiv 10 \bmod 11. \end{array} \tag{11}$$

□

Let $p > 2$ be a prime number, and consider now the equation

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = 0. \tag{12}$$

Dividing $\Phi_p(x)$ by $x^{(p-1)/2}$ and using the change of variable $y = x + 1/x$, we obtain

$$\Psi_{(p-1)/2}(y) = \frac{\Phi_p(x)}{x^{(p-1)/2}} = 0, \tag{13}$$

which is an equation of degree $(p-1)/2$. Let ζ be a primitive pth root of 1, that is, let $\zeta = e^{2\pi i/p}$. Then the roots of (13) take on the form $\eta_j = \zeta^j + \frac{1}{\zeta^j} = \zeta^j + \zeta^{p-j}$, where $j = 1, \ldots, \frac{p-1}{2}$. Equation (10) yields that $\eta_j = \zeta^{g^{\vartheta(j)}} + \zeta^{g^{\vartheta(p-j)}}$. The rule of assignment

$$\zeta^{g^0} \mapsto \zeta^{g^1} \mapsto \zeta^{g^2} \mapsto \cdots \mapsto \zeta^{g^{p-2}} \mapsto \zeta^{g^0} \tag{14}$$

implies now that $\zeta^{g^{\vartheta(j)}} + \zeta^{g^{\vartheta(p-j)}} \mapsto \zeta^{g^{\vartheta(j)+1}} + \zeta^{g^{\vartheta(p-j)+1}}$ which, in turn, induces a permutation on the roots $\eta_1, \eta_2, \ldots, \eta_{(p-1)/2}$, namely,

$$\eta_1 \mapsto \eta_{\sigma(1)} \mapsto \eta_{\sigma(2)} \mapsto \cdots \mapsto \eta_{\sigma((p-1)/2)} \mapsto \eta_1. \tag{15}$$

Example 5. If $\zeta = e^{2\pi i/11}$, then $\eta_1 = \zeta + \zeta^{10}$, $\eta_2 = \zeta^2 + \zeta^9$, $\eta_3 = \zeta^3 + \zeta^8$, $\eta_4 = \zeta^4 + \zeta^7$, and $\eta_5 = \zeta^5 + \zeta^6$. Using (11) and (14) we deduce that

$$\begin{array}{rcl} \eta_1 & = & \zeta^{2^0} + \zeta^{2^5} \mapsto \zeta^{2^1} + \zeta^{2^6} = \eta_2, \\ \eta_2 & = & \zeta^{2^1} + \zeta^{2^6} \mapsto \zeta^{2^2} + \zeta^{2^7} = \eta_4, \\ \eta_3 & = & \zeta^{2^8} + \zeta^{2^3} \mapsto \zeta^{2^9} + \zeta^{2^4} = \eta_5, \\ \eta_4 & = & \zeta^{2^2} + \zeta^{2^7} \mapsto \zeta^{2^3} + \zeta^{2^8} = \eta_3, \\ \eta_5 & = & \zeta^{2^4} + \zeta^{2^9} \mapsto \zeta^{2^5} + \zeta^{2^{10}} = \zeta^{2^5} + \zeta^{2^0} = \eta_1. \end{array}$$

Comparing these assignments with (8), we readily check that this is the same permutation discovered by Vandermonde. □

Let $\omega = e^{4\pi i/(p-1)}$ be a primitive $[(p-1)/2]$th root of 1, and let

$$V_k\left(\eta_1, \ldots, \eta_{\frac{p-1}{2}}\right) = \sum_{j=1}^{(p-1)/2} \omega^{(j-1)k} \eta_j,$$

for each $k = 1, \ldots, \frac{p-1}{2}$, be the Lagrange resolvent. The permutation (15) leaves the following function (actually, a polynomial in the variable ω) invariant:

$$V_k\left(\eta_1, \eta_2, \ldots, \eta_{\frac{p-1}{2}}\right)^{(p-1)/2}.$$

From Lagrange's formula (4), it follows that

$$\eta_j = \frac{2}{p-1}\left(\sum_{k=1}^{(p-1)/2} w^{-(j-1)k} \sqrt[(p-1)/2]{V_k\left(\eta_1, \eta_2, \ldots, \eta_{\frac{p-1}{2}}\right)^{(p-1)/2}}\right),$$

which shows that the roots of (12) are expressible in terms of radicals if ω is. We conclude that the roots η_j can be expressed through radicals.

Gauss employed induction to prove that the roots of $x^n = 1$ can be expressed by means of radicals. The cases $n = 1, 2$ being immediate, one supposes that the result is true for $k < n$. If n is not prime, then $n = uv$, for suitable integer numbers u and v. Let $x_1, \dots, x_u$ and $y_1, \dots, y_v$ be the uth and vth roots of 1, respectively. Then, $x_j \sqrt[u]{y_k}$, for $j = 1, \dots, u$ and $k = 1, \dots, v$, are the roots of $x^n = 1$. Thus, we can suppose that n is a prime number, in which case, the conclusion is reached from the previous discussion in view that, by hypothesis, $\omega = e^{4\pi i/(p-1)}$ is a primitive $[(p-1)/2]$th root of 1, so expressible in terms of radicals. An excellent reference for a rigorous proof of Gauss method is [26]

6. Closing remarks

Let p and q be positive integers. The division algorithm states that there exist positive integers k and r, such that $p = kq + r$ and $0 \leq r < q$. Using the trigonometric identities

$$\begin{aligned} \sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}, \\ \cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}, \\ \cos(\theta+\phi) &= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi), \end{aligned}$$

we obtain

$$\sin\left(\frac{p\pi}{q}\right) = \pm\sqrt{\frac{1-\cos\left(\frac{2r}{q}\pi\right)}{2}}, \quad \cos\left(\frac{p\pi}{q}\right) = \pm\sqrt{\frac{1+\cos\left(\frac{2r}{q}\pi\right)}{2}}.$$

Since all of the nth roots of 1 are given by de Moivre's formula and they can be expressed using radicals, then $\sin(p\pi/q)$ and, thus, also $\cos(p\pi/q)$, can be written by means of radicals.

Notice that the solutions of Equation $x^n = 1$ may be represented as the n vertices of a regular polygon. Using this fact, Gauss [7] concluded that a regular polygon with n vertices is constructible with ruler and compass if n is of the form $2^m p_1 p_2 \cdots p_k$, where the numbers $p_1, p_2, \dots, p_k$ are different prime numbers, and each of them assumes the form $2^{2^j} + 1$ (that is, Fermat's primes [4]). The converse is due to Pierre Laurent Wantzel (1814–1848) [17]. In particular, it is impossible to construct a regular polygon of 7, 9, 11 and 13 vertices, using ruler and compass.

As we mentioned in Section 2, Abel showed that there does not exist a general formula for the quintic, but it was Évariste Galois who provided the complete solution to the problem of solubility of equations through radicals. More precisely, given a polynomial equation, Galois associates a unique algebraic object, namely, its *Galois group*, and establishes that the equation is soluble by means of radicals if and only if the Galois group is soluble [26].

To close this work, the authors would like to state that they have performed an exhaustive research on available methods and approaches to try to solve the problem under consideration. Instead, they have acquired some historical perspective of the problem, and the present manuscript is a summary of what they consider relevant developments toward its satisfactory solution. What began as an undergraduate project (to prove that $\sin(p\pi)$ and $\cos(p\pi)$ is expressible in terms of radicals for $p \in \mathbb{Q}$), turned out to be a more general task in which the authors have not been able to elucidate the way to apply certain historical and traditional approaches. Remark 2 raises the following open question.

Problem 6. Characterize those $p \in \mathbb{R}$ such that $\sin(p\pi)$ and $\cos(p\pi)$ have a representation in terms of radicals.

At the closure of the present manuscript, the authors have learned through private communications that G. Villa and M. Rzedowski, professors of the Department of Automatic Control at the Centro de Investigación y de Estudios Avanzados (CINVESTAV), Mexico, have successfully established the existence of an infinite number of angles θ for which $\sin\theta$ and $\cos\theta$ cannot be expressed through radicals. However, to the best of our knowledge, Problem 6 has not been solved yet.

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