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ON THE COEFFICIENTS OF THE POLYNOMIAL INDUCED BY RECIPROCAL ROOTS REVISITED

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ABSTRACT. Consider a polynomial P with complex coefficients and nonzero roots. What is the relation between the coefficients of P, and those of the polynomial $\overline{P}$ whose roots are reciprocal of the roots of P? Is it possible to express the coefficients of $\overline{P}$ through a formula which depends on the coefficients of P? The purpose of this note is to revisit this problem and respond affirmatively to these questions. Some examples will be provided for illustration purposes and as a motivation for this communication.

Let us consider a polynomial P with coefficients in the complex numbers. For the sake of concreteness, let us assume that $P(z) = 2 + 3z + z^2$. This polynomial has $r_1 = -1$ and $r_2 = -2$ as its only roots. Define $\overline{r}_i$ as the reciprocal of r_i, for each $i = 1, 2$. Obviously, $\overline{r}_1 = -1$ and $\overline{r}_2 = -\frac{1}{2}$, and it is easy to check that the polynomial with roots $\overline{r}_1$ and $\overline{r}_2$ is given by $\overline{P}(z) = \frac{1}{2} + \frac{3}{2}z + z^2$. A natural question is whether there exists a relation between the coefficients of P and those of $\overline{P}$. The purpose of the present note is to respond this question affirmatively. To that end, we will revisit a result from the theory of polynomials, and we will establish it in its most general form, which will include the case when 0 is a root of the polynomial P.

For the remainder of this communication, P will represent a non-constant polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $a_0, a_1, \ldots, a_n \in \mathbb{C}$. Recall that the *degree* of P is the largest integer $N \in \mathbb{N} \cup \{0\}$ such that $a_N \neq 0$. In this case, a_N is the *leading coefficient* of P. A *root* of the polynomial P is some $r \in \mathbb{C}$ for which $P(r) = 0$. If $r \in \mathbb{C}$ is a root of P, then we say that it has *multiplicity* $m \in \mathbb{N}$ if

$$P(z) = (z - r)^m Q(z),$$

where Q is a complex polynomial such that $Q(r) \neq 0$.

It is well known that non-constant polynomials with coefficients in the complex numbers possess complex roots. More precisely, the following is a standard result in algebra and the theory of polynomials.

THEOREM 1 (Fundamental Theorem of Algebra [2]). *If P is a polynomial of degree $n \in \mathbb{N}$ with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$, then the polynomial equation*

$$P(z) = \sum_{j=0}^{n} a_j z^j = 0$$

has at least one solution $z \in \mathbb{C}$. Moreover, there exists exactly n complex numbers $r_1, \ldots, r_n$, such that $P(z) = a_n(z - r_1) \cdots (z - r_n)$.

In order to prove the main result of this work, we will need to make use of elementary symmetric polynomials. Recall that the k-th *elementary symmetric polynomial* in n complex variables $z_1, \ldots, z_n$ is the polynomial $S_k : \mathbb{C}^n \to \mathbb{C}$ given by

$$S_k(z_1, \ldots, z_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} z_{j_1} \cdots z_{j_k},$$

where $1 \leq k \leq n$ and $S_0(z_1, \ldots, z_n) = 1$ (see [1]).

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The following technical result will be required in the sequel. The proof of this result is based on Proposition V.8.4 in [1].

LEMMA 2. *Let P be a polynomial of degree $n \in \mathbb{N}$, with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and roots $r_1, \ldots, r_n \in \mathbb{C}$. If $0 \leq k \leq n$, then*

$$S_{n-k}(r_1, \ldots, r_n) = (-1)^{n-k} \frac{a_k}{a_n}.$$

Proof. We know that $P(z) = a_n(z - r_1)(z - r_2) \cdots (z - r_n)$ from the Fundamental Theorem of Algebra. So, it suffices to prove that the following equation holds:

$$P(z) = a_n \sum_{j=0}^{n} (-1)^{n-j} S_{n-j}(r_1, \ldots, r_n) z^j, \quad \forall z \in \mathbb{C}. \tag{1}$$

To do this, we expand $(z - r_1)(z - r_2) \cdots (z - r_n)$ algebraically. The result is a sum of terms of the form $t_1 t_2 \cdots t_n$, where $t_j = z$ of $t_j = -r_j$, for each $j = 1, \ldots, n$. Let $1 \leq j \leq n$, and consider a term $t_1 t_2 \cdots t_n$ in which $t_i = -r_i$ appears exactly j times. It follows that $t_1 t_2 \cdots t_n = (-1)^j r_{i_1} r_{i_2} \cdots r_{i_j} z^{n-j}$, where $1 \leq i_1 < i_2 < \ldots < i_j \leq n$. When adding together all of these terms, we obtain $(-1)^j S_j(r_1, \ldots, r_n) z^{n-j}$, for each $1 \leq j \leq n$, whence it follows that

$$P(z) = a_n \sum_{j=0}^{n} (-1)^j S_j(r_1, \ldots, r_n) z^{n-j}.$$

Equation (1) is reached now after a change of variable. □

The following is the main result of this work.

THEOREM 3. *Let P be a polynomial of degree $n \in \mathbb{N}$, coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and roots $r_1, \ldots, r_n \in \mathbb{C}$. Define the complex polynomial $\overline{P}$ as follows:*

(1) *The degree of $\overline{P}$ is equal to n, and a_n is its leading coefficient.*
(2) *If $r_i \neq 0$, then $\overline{r}_i := \frac{1}{r_i}$ is one of its roots.*
(3) *If $r_i = 0$, then $\overline{r}_i := r_i$ is one of its roots.*

If m is the number of roots equal to zero, then

$$\overline{P}(z) = a_n \sum_{j=0}^{n-m} \frac{a_{n-j}}{a_m} z^{m+j}.$$

This polynomial will be called the polynomial induced by the reciprocal roots *of P, and it i unique up to the multiplication by a nonzero complex constant.*

Proof. Let $b_0, b_1, \ldots, b_n$ be the coefficients of the polynomial $\overline{P}$, where b_n is the leading coefficient. The proof will consider two cases.

- **Case 1.** Assume firstly that all the roots are nonzero. Hence,

$$\begin{aligned} S_{n-k}(\overline{r}_1, \ldots, \overline{r}_n) &= \sum_{1 \leq j_1 < \cdots < j_{n-k} \leq n} \overline{r}_{j_1} \cdots \overline{r}_{j_{n-k}} \\ &= \sum_{1 \leq j_1 < \cdots < j_{n-k} \leq n} \frac{1}{r_{j_1}} \cdots \frac{1}{r_{j_{n-k}}}. \end{aligned}$$

To calculate the right-hand side of this equation observe that

$$\begin{aligned} S_{n-k}(\overline{r}_1, \ldots, \overline{r}_n) &= \frac{1}{r_1 \cdots r_n} \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} r_{j_1} \cdots r_{j_k} \\ &= \frac{S_k(r_1, \ldots, r_n)}{S_n(r_1, \ldots, r_n)}. \end{aligned}$$

The conclusion in this case is reached now using Lemma 2 and noticing that

$$b_k = a_n(-1)^{n-k}\frac{S_k(r_1,\dots,r_n)}{S_n(r_1,\dots,r_n)} = a_n\frac{a_{n-k}}{a_0},$$

for each $0 \le k \le n$.

- **Case 2.** Suppose now that there are $0 < m < n$ roots which are equal to zero. Without lost generality, assume that $\overline{r}_1, \overline{r}_2, \dots, \overline{r}_q$ are the nonzero roots, where $q = n - m$. Since zero has multiplicity equal to m, then there is a complex polynomial Q of degree q with leading coefficient equal to 1, such that $Q(0) \neq 0$. If $c_0, \dots, c_n$ are the coefficients of Q, then $a_{m+j} = a_n c_j$, for each $0 \le j \le q$. Moreover, if $\overline{Q}$ is the polynomial induced by the reciprocal roots of Q, then

$$\overline{P}(z) = a_n z^m \overline{Q}(z) = a_n \sum_{j=0}^{q}(-1)^{q-j} S_{q-j}(\overline{r}_1,\dots,\overline{r}_q) z^{m+j},$$

where the roots of $\overline{Q}$ are $\overline{r}_1, \overline{r}_2, \dots, \overline{r}_q$. Applying the first result to Q, then

$$\overline{P}(z) = a_n \sum_{j=0}^{q} \frac{c_{q-j}}{c_0} z^{m+j} = a_n \sum_{j=0}^{n-m} \frac{a_{n-j}}{a_m} z^{m+j}, \quad \forall z \in \mathbb{C}.$$

Notice finally that (1) holds for $m = n$. □

It is important to point out a convenient simplification in the expression of the polynomial $\overline{P}$ in the conclusion of Theorem 3 when none of the roots of P is equal to zero. The result is provided in the next corollary.

Corollary 4. *Let P be a polynomial of degree $n \in \mathbb{N}$, with complex coefficients $a_0, a_1, \dots, a_n$ and roots $r_1, \dots, r_n \in \mathbb{C} \setminus \{0\}$. Then a polynomial whose roots are the reciprocals of $r_1, \dots, r_n$ is given by*

$$Q(z) = a_n + a_{n-1}z + \dots + a_0 z^n.$$

Proof. Under the hypotheses of this corollary, zero is a root of multiplicity $m = 0$. Theorem 3 implies that the polynomial induced by the reciprocal roots of P is

$$\overline{P}(z) = \frac{a_n}{a_0}\sum_{j=0}^{n} a_{n-j} z^j. \tag{2}$$

Observe that a_n/a_0 is a nonzero constant. This implies that $Q(z) = \frac{a_0}{a_n}\overline{P}(z)$ has the same roots as $\overline{P}$, whence the conclusion of this result readily follows. □

Before we close this work, we provide a couple of applications of Theorem 3.

Example 5. Consider the polynomial $P(z) = 2 + z + 2z^2 + z^3$, whose roots are $r_1 = -2$, $r_2 = i$ and $r_3 = -i$. Observe that the reciprocal roots are $\overline{r}_i = -\frac{1}{2}$, $\overline{r}_2 = -i$ and $\overline{r}_3 = i$. It follows that the polynomial induced by the reciprocal roots of P is

$$\overline{P}(z) = \frac{1}{2} + z + \frac{1}{2}z^2 + z^3.$$

On the other hand, following the notation used in this work, we have $n = 3$, $a_0 = 2$, $a_1 = 1$, $a_2 = 2$ and $a_3 = 1$. Moreover, zero is a root of multiplicity $m = 0$ for P. The conclusion of Theorem 3 states that

$$\overline{P}(z) = \frac{a_3}{a_0}\sum_{j=0}^{3} a_{3-j} z^j = \frac{1}{2}\left(1 + 2z + z^2 + 2z^3\right).$$

Obviously, both results for $\overline{P}(z)$ agree in this example. Moreover, multiplying $\overline{P}(z)$ by the nonzero constant 2, we obtain that $Q(z) = 1 + 2z + z^2 + 2z^3$ is a polynomial whose roots are the reciprocal of the roots of $P(z)$. This result agrees with the conclusion of Corollary 4, as expected.

Example 6. We consider next the polynomial $P(z) = z^2 + z^4$, which has 0, i and $-i$ as its only roots. Moreover, zero is a root of multiplicity $m = 2$. In this example, the coefficients of P are $a_0 = a_1 = a_3 = 0$ and $a_2 = a_4 = 1$. According to Theorem 3, the polynomial induced by the reciprocal roots of P is

$$\overline{P}(z) = a_4 \sum_{j=0}^{n} \frac{a_{4-j}}{a_2} z^{2+j} = z^2 + z^4.$$

On the other hand, we can calculate the polynomial whose roots are the reciprocals of i and $-i$, and which has 0 as a root of multiplicity 2. It is clear that the result agrees with the polynomial $\overline{P}(z)$, as expected.

References

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