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A NEW PROOF OF IONESCU-WEITZENBÖCK'S INEQUALITY, USING A LINEAR MAP

MARTIN CELLI

ABSTRACT. Using a linear map, we give a new proof of Ionescu-Weitzenböck's inequality.

This note was inspired by the *miniaturas* of the Gaceta de la Real Sociedad Matemática Española and the *mathbits* of the American Mathematical Monthly. Such a format offers short, elementary, deep and original demonstrations, which can be understood and enjoyed without specific knowledge.

THEOREM. *For any triangle ABC with area $\mathcal{A}$, we have:*

$$BC^2 + CA^2 + AB^2 \geq 4\sqrt{3}\mathcal{A}\cdot$$

Proof. Let us consider the linear map

$$T(\vec{U}) = (\overrightarrow{BC} \cdot \vec{U})\overrightarrow{BC} + (\overrightarrow{CA} \cdot \vec{U})\overrightarrow{CA} + (\overrightarrow{AB} \cdot \vec{U})\overrightarrow{AB}\cdot$$

We have:

$$\begin{cases} T(\overrightarrow{BC}) = BC^2\overrightarrow{BC} + (\overrightarrow{CA} \cdot \overrightarrow{BC})\overrightarrow{CA} - (\overrightarrow{AB} \cdot \overrightarrow{BC})(\overrightarrow{BC} + \overrightarrow{CA}) \\ T(\overrightarrow{CA}) = (\overrightarrow{BC} \cdot \overrightarrow{CA})\overrightarrow{BC} + CA^2\overrightarrow{CA} - (\overrightarrow{AB} \cdot \overrightarrow{CA})(\overrightarrow{BC} + \overrightarrow{CA}) \end{cases}$$

This gives us the matrix of T in the basis $(\overrightarrow{BC}, \overrightarrow{CA})$:

$$\begin{pmatrix} BC^2 - \overrightarrow{AB} \cdot \overrightarrow{BC} & (\overrightarrow{BC} - \overrightarrow{AB}) \cdot \overrightarrow{CA} \\ (\overrightarrow{CA} - \overrightarrow{AB}) \cdot \overrightarrow{BC} & CA^2 - \overrightarrow{AB} \cdot \overrightarrow{CA} \end{pmatrix} = \begin{pmatrix} 2BC^2 + P & CA^2 + 2P \\ BC^2 + 2P & 2CA^2 + P \end{pmatrix},$$

denoting $P = \overrightarrow{BC} \cdot \overrightarrow{CA}$. Thus, we can compute the trace and the determinant of T:

$$\begin{cases} \operatorname{tr}(T) = BC^2 + CA^2 + AB^2 \\ \det(T) = 3(BC^2CA^2 - P^2) = 3(\det(\overrightarrow{BC}, \overrightarrow{CA}))^2 = 12\mathcal{A}^2 \end{cases}$$

As it is symmetric $(T(\vec{U}) \cdot \vec{V} = T(\vec{V}) \cdot \vec{U})$, the map T es real diagonalizable. So the discriminant of its characteristic polynomial $x^2 - \operatorname{tr}(T)x + \det(T)$ is positive:

$$(\operatorname{tr}(T))^2 \geq 4\det(T),$$

$$(BC^2 + CA^2 + AB^2)^2 \geq 48\mathcal{A}^2 = (4\sqrt{3}\mathcal{A})^2\cdot$$

□

Ionescu-Weitzenböck's inequality was published by I. Ionescu in 1897, in the problems section of the Romanian Mathematical Gazette, and by R. Weitzenböck in 1919. It was also one of the problems to solve at the Third International Mathematical Olympiad in 1961. Many proofs of this inequality have been discovered, all of them are based on arguments different from ours. Some proofs can be found in the references below: visual proofs ([1], [4]), geometric proofs ([2], [3], [5], [6]), vector proofs ([3], [4]). New proofs of classic geometric inequalities regularly inspire tools for more advanced results.

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Martin Celli
Universidad Autónoma Metropolitana,
Unidad Iztapalapa,
División de Ciencias Básicas e Ingeniería,
Departamento de Matemáticas.
Av. Ferrocarril San Rafael Atlixco 186, Col. Leyes de Reforma 1^a Sección,
Alcaldía Iztapalapa, C.P. 09310 CDMX, México.
e-mail: `cell@xanum.uam.mx`