MatCont tutorial on starting up homoclinic orbits from a Bogdanov-Takens point

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Abstract

MatCont can start the continuation of a branch of saddle-homoclinic orbits from a given BT point. We update the BT_Hom initializer using the results in a recent study that improves on the previous literature. We propose to use an extra parameter (`kfactor`) which allows users to better manipulate the initialization of the homoclinc continuation. Using the new intializer, we start branches of homoclinic orbits in multi-dimensional models and we point out the importance of the new parameter. In MatCont5.3 where the new initializer is first introducted, `kfactor` is hard-coded in the file init_BT_Hom where the user may have to change it.

1 Introduction

The study of the homoclinic orbits that appear near singular points of codimension 2 such as Bogdanov-Takens (BT) is a part of global bifurcation theory, see [16, 22, 2, 3, 19]. Consider the following system of ordinary differential equations (ODE's)

$$\dot{y} = f(y, \alpha), \tag{1}$$

where $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^m$. We say that a solution $y(t)$ of (1) is a homoclinic orbit at $\alpha = \alpha_0$ if

$$y(t) \to y_0 \quad \text{as} \quad t \to \pm\infty, \quad y(0) \neq y_0 \tag{2}$$

and

$$f(y_0, \alpha_0) = 0. \tag{3}$$

There are two types of homoclinic orbits with codimension 1, namely homoclinic to hyperbolic saddle (HHS), if y_0 is a saddle, and homoclinic to saddle node (HSN), if y_0 is a saddle-node. Codimension 1 implies that in generic dynamical systems with two free parameters these orbits exist along curves in the parameter plane [15].

The BT bifurcation occurs if the equilibrium of (1) has a zero eigenvalue of algebraic multiplicity two and geometric multiplicity one. The homoclinic orbits near a BT point are of the saddle type (HHS).

The idea of starting homoclinic orbits numerically from a BT point in planar systems was developed by Rodriguez L. *et al.* [22]. This idea was based on *shooting* (numerical integration of orbits in the stable and unstable directions of the equilibria). Beyn W. [2] treated the general n-dimensional problem. He investigated the problem of the homoclinic orbits that emanate from a BT point as a branch switching problem and he used boundary value methods (*BVMs*) to compute homoclinic orbits. Such methods truncate the problem to a finite time interval and impose projection boundary conditions at the end points of the interval. In the BT case, since y_0 is hyperbolic, the truncated boundary value problem is formally well-posed. Moreover, the existence of a homoclinic solution to the original problem on the infinite interval implies the existence of a solution to the truncated problem [1].

To continue homoclinic orbits using *BVMs*, we need an initial point with the corresponding homoclinic solution. MatCont [12] originally used a method adopted from [3] to derive such initial solution. However,

this method needs correction. In [20] the method was improved and a better homoclinic predictor was derived.

In the present tutorial we show how the new intializer can be used in multi-dimensional systems.

The paper is organized as follows. Section 2 is a review on the main mathematical concepts. In section 3 we describe the algorithm used in MatCont to initialize the homoclinic continuation from a computed BT point. Numerical examples with MatCont are in section 4. Section 5 contains conclusions and suggestions for users.

2 The homoclinic prediction

Center manifold theory provides a method to reduce the dimensionality of (1) to the number of zero eigenvalues. In a BT neighborhood, we can reduce the nD system (1) to a 2D system. Details and extensive theoretical studies are given in [25, 16, 19].

Normal form theory allows us to simplify the dynamics in the center manifold by eliminating all unnecessary nonlinear terms which do not affect the existence of stability and solutions. A normal form of (1) at the BT point is given by

$$\begin{aligned} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= a w_0^2 + b w_0 w_1 + \mathbb{O}(\|w\|^3), \end{aligned} \tag{4}$$

where $(w_0, w_1)^T \in \mathbb{R}^2$ and $ab \neq 0$ (the non-degeneracy condition). We call a and b the BT normal form coefficients. Guckenheimer and Holmes [16] showed that the family

$$\begin{aligned} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 + \mathbb{O}(\|w\|^3 + \|\beta\|\|w\|^2), \end{aligned} \tag{5}$$

represents a parameter-dependent normal form of (4). By a suitable blowup transformation, we can construct an approximation for the homoclinic orbits in the BT normal form (5) and then transfer system (5) with the resulted homoclinic approximation to the original equation (1). For the literature we refer, in particular, to [20, 22, 2, 3]. So, we arrive at the homoclinic predictor for the original system (1)

$$\begin{aligned} \alpha &= \alpha_0 + \varepsilon^2 \gamma_1 + \varepsilon^4 \gamma_2, \\ y(t) &= y_0 + \varepsilon^2 g_1(\varepsilon t) + \varepsilon^3 g_2(\varepsilon t). \end{aligned} \tag{6}$$

Here γ_1, γ_2, $g_1(\varepsilon t)$, and $g_2(\varepsilon t)$ are defined in [20], where ε is called the initial amplitude of the homoclinic orbit. In MatCont ε is chosen by the user.

3 Continuation of homoclinic orbits

To generate the initial data to start the continuation of homoclinic orbits from a BT point, we use a procedure that consists of several steps.

First, the condition (2) is defined on an infinite time interval $(-\infty, \infty)$. This interval has to be truncated to a finite interval with suitable boundary conditions, say $[-T, +T]$, where T is the half-return time. So, after applying this truncation of time, equation (1) becomes [1]

$$\dot{x} - 2T f(x(t), \alpha) = 0. \tag{7}$$

The interval $[-T, +T]$ is rescaled to $[0, 1]$ and divided into mesh intervals where the solution is approximated by a vector polynomial (represented via the interpolation formula depending on *Lagrange* basis polynomials). The mesh is non-uniform and adaptive. Each mesh interval is further subdivided by equidistant fine mesh

[1] We will reserve x for system (1) with time truncation and rescaling.

points. Also, each mesh interval contains m collocation points (*Gauss* points). These points are the rescaled roots of the $m^{\text{th}}-$degree *Legendre* polynomials [19, 5]. The numbers ntst of mesh intervals and ncol = m of collocation points are part of the continuation data chosen by the user. Equation (7) must be satisfied at each collocation point. An initial approximation for the homoclinic solution that satisfies (1), (2) and (3) at some parameter value α is computed by the homoclinic predictor (6). We choose the initial T such that, at the end points, the distance between y_0 and $y(\pm T)$ is sufficiently small [2], say δ_0, where

$$\delta_0 = \frac{\|y(0) - y_0\|}{\kappa}, \tag{8}$$

and κ is a code parameter (`kfactor`). This leads to an important connection between the initial amplitude value ε, the code parameter κ and the initial T.

Second, the distance between $x(0)$ (respectively, $x(1)$) and x_0 must be monitored. We define

$$\begin{aligned} \varepsilon_0 =& \|x(0) - x_0\| \\ \varepsilon_1 =& \|x(1) - x_0\|. \end{aligned} \tag{9}$$

The initial values for ε_0 and ε_1 are determined by the predictor (6). The half-return time T, ε_0 and ε_1 are called the homoclinic parameters. We note that either one or two homoclinic parameters can be free.

Third, if two homoclinic parameters are free, then the following integral phase condition is added

$$\int_0^1 \left\langle x(t) - \tilde{x}(t), \dot{\tilde{x}}^{\mathrm{T}}(t) \right\rangle \, \mathrm{d}t = 1, \tag{10}$$

where $\tilde{x}$ is the homoclinic solution obtained at the previously found point on the curve. Equation (10) is a necessary condition for a local minimum of the l_2-distance between $x(t)$ and $\tilde{x}(t)$ over time shifts [19].

Fourth, the boundary value problem (7), (3) and (10) needs suitable boundary conditions at 0 and 1 (i.e., replace (2) by suitable boundary conditions). The following projection boundary conditions are used to solve this problem

$$\begin{aligned} Q_S(x(0) - x_0) &= 0, \\ Q_U(x(1) - x_0) &= 0, \end{aligned} \tag{11}$$

where Q_S and Q_U are the $n_s \times n$ and $n_u \times n$ matrices whose rows form a basis for the stable and unstable eigenspace respectively of $A^T(x_0, \alpha)$. The projection boundary conditions place the solution at the two end points in the unstable and stable eigenspace of $A(x_0, \alpha)$ respectively [8]. The MatCont software [11], uses specific algorithm dependent on the real Schur factorization to construct Q_S and Q_U (see [9, 4, 10, 14, 13, 18]).

4 Examples

We will use MatCont to start homoclinic orbits that emanate from a given BT point.

4.1 Normal form

Consider the BT normal form system (5) without the order terms, the bifurcation analysis of which was presented in [16], [25] and [22]. We use MatCont to start a continuation of equilibria with fixed normal form coefficients $(a, b) = (-1, 1)$, initial parameter values $(\beta_1,\ \beta_2) = (1, -2)$ and the equilibrium point $(w_0, w_1) = (1, 0)$; β_1 is the free parameter. Two bifurcation points are detected along the curve of equilibria, namely a limit point (LP) and Hopf (H). The LP continuation is now carried out with free β_1, β_2 to detect a BT bifurcation point at $(\beta_1, \beta_2) = (0, 0)$. We start the BT_Hom initializer with β_1 and β_2 as the free system parameters. In the MatCont **Starter** window, choose ε_0 and ε_1 as the free homoclinic parameters. With 40 test intervals (ntst), 4 collocation points (ncol), the amplitude (ε) equal to 0.05 and `kfactor` = 20 one can start the homoclinic curve. It turns out that $T = 70.806$. Repeat the continuation steps using different signs for a and b (i.e., $(a, b) = (1, 1)$, $(a, b) = (1, -1)$ and $(a, b) = (-1, -1)$). In Fig.1 and Fig.2 we present the computed homoclinic orbits in parameter and phase space, respectively.

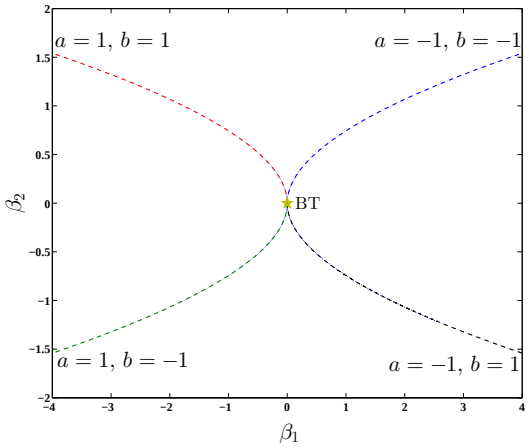


Figure 1: Homoclinic orbits in parameter (β_1, β_2)

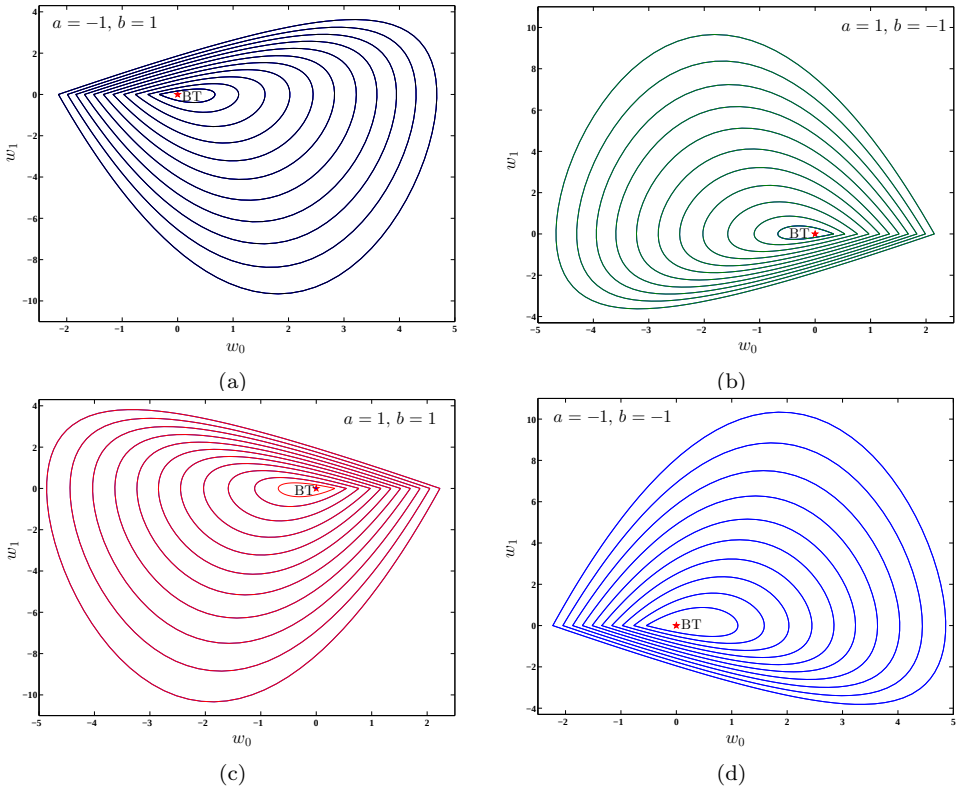


Figure 2: Homoclinic orbits in state (w_0, w_1) space.

4.2 Predator-Prey systems with constant rate harvesting

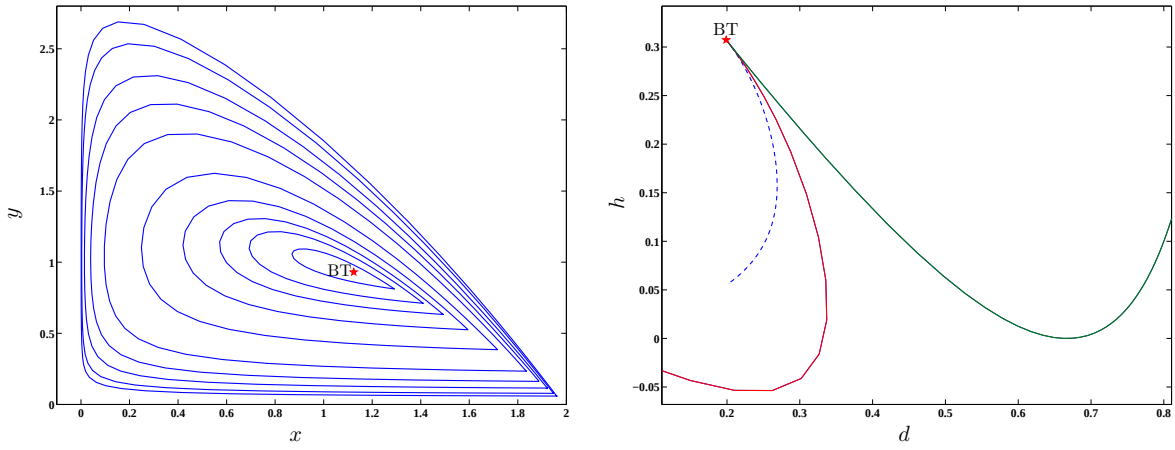
Consider the following predator-prey system with constant rate predator harvesting [6]:

$$\begin{aligned} \dot{x} &= rx(1 - \frac{x}{k}) - \frac{yx}{e + x}, \\ \dot{y} &= y(-d + \frac{x}{e + x}) - h, \end{aligned} \tag{12}$$

where k is the carrying capacity of the prey population, d is the death rate of the predator, r is the intrinsic growth rate of the prey population, and h is the harvesting rate. Xiao and Ruan [26] show the existence of a BT bifurcation in system (12) and sketch the global bifurcation diagram including the homoclinic curve which emanates from the BT point. We study the occurrence of homoclinic orbits that emanate from the computed BT point using MatCont. We fix the parameter values as follows: $r = 1$, $e = 1$, $k = 2$, $h = 0.5$, $d = 0$, then we compute the equilibria with free parameter d and initial value for state variables $x = 1.1$ and $y = 0.09$. A limit point (LP) is detected. Compute the fold curve passing through it with (d, h) as free parameters. One BT point is detected (see Table 1). To initialize the homoclinic orbit from the BT point with (d, h) free system parameters, choose `kfactor` $= 2$ and in the MatCont **Starter** window input the following numerical data: $\varepsilon = 0.001$, ntst $= 20$, ncol $= 4$ and in the **Continuer** window set Adapt $= 1$. Activate ε_0 and ε_1 as a free homoclinic parameters and then **Compute - Backward**. The result should be as in Fig.3. Notice that during continuation $T = 1416.35$.

Table 1: Parameter, state and BT normal form coefficient (a, b) values at the bifurcation points in Fig.3.

Label	d	h	State variables	Normal form coefficients (a, b)
BT	0.198	0.307	$(1.12, 0.93)$	$(-0.18, 0.38)$



(a) . (b) .

Figure 3: (a) The homoclinic orbits in state space for system (12), (b) The homoclinic orbit in parameter space. The dashed blue curve is the homoclinic curve. The red is the Hopf curve and the green is the LP curve.

4.3 CO-oxidation in a platinum model.

Consider the following chemical model which describes CO-oxidation in platinum [17, 7, 2]

$$\begin{aligned} z &= 1 - x - y - s, \\ \dot{x} &= 2k_1 z^2 - 2k_{-1} x^2 - k_3 xy, \\ \dot{y} &= k_2 z - k_{-2} y - k_3 xy, \\ \dot{s} &= k_4 (z - \lambda s). \end{aligned} \tag{13}$$

where $\lambda = \frac{k_{-4}}{k_4}$. The underlying reaction scheme is displayed in [7] and we notice that a factor 2 is missing in front of $k_1 z^2$ in [17]. The reaction rates (constants) above are fixed as follows $k_1 = 2.5$, $k_{-1} = 1$,

$k_3 = 10$, $k_{-2} = 0.1$, $k_4 = 0.0675$, $k_2 = 1.4707$, $\lambda = 0.4$. Compute the equilibria curve with free k_2 and initial state variables $x = 0.00295$, $y = 0.76211$, $s = 0.1678$. Two limit points (LP) are detected. Start the LP continuation from any of the two LP-points with (λ, k_2) free system parameters. Two BT points are detected, see Table 2. From BT_1 point and with $(k_2,\ \lambda)$ as free system parameters start the homoclunic curve continuation using `kfactor` $= 20$, $\varepsilon = 1 \times 10^{-4}$, ntst $= 80$, ncol $= 4$, Adapt $= 1$, in the continuer window set the **InitStepsize** $= 0.001$ and **Compute** - **Backward** with ε_0, ε_1 as free homoclinic parameters and the fixed $T = 29821.19$. For BT_2 use $\varepsilon = 1 \times 10^{-5}$, `kfactor` $= 50$, the fixed $T = 494129.24$ and then **Compute** - **Backward**. The results are presented in Fig.4

Table 2: Parameter, state and BT normal form coefficient (a, b) values at the bifurcation points in Fig.4.

Label	k_2	λ	State variables	Normal form coefficients (a, b)
BT_1	1.42	0.97	$(0.12, 0.32, 0.29)$	$(-0.08, -2.14)$
BT_2	1.162	0.72	$(0.02, 0.64, 0.20)$	$(-0.05, -1.94)$

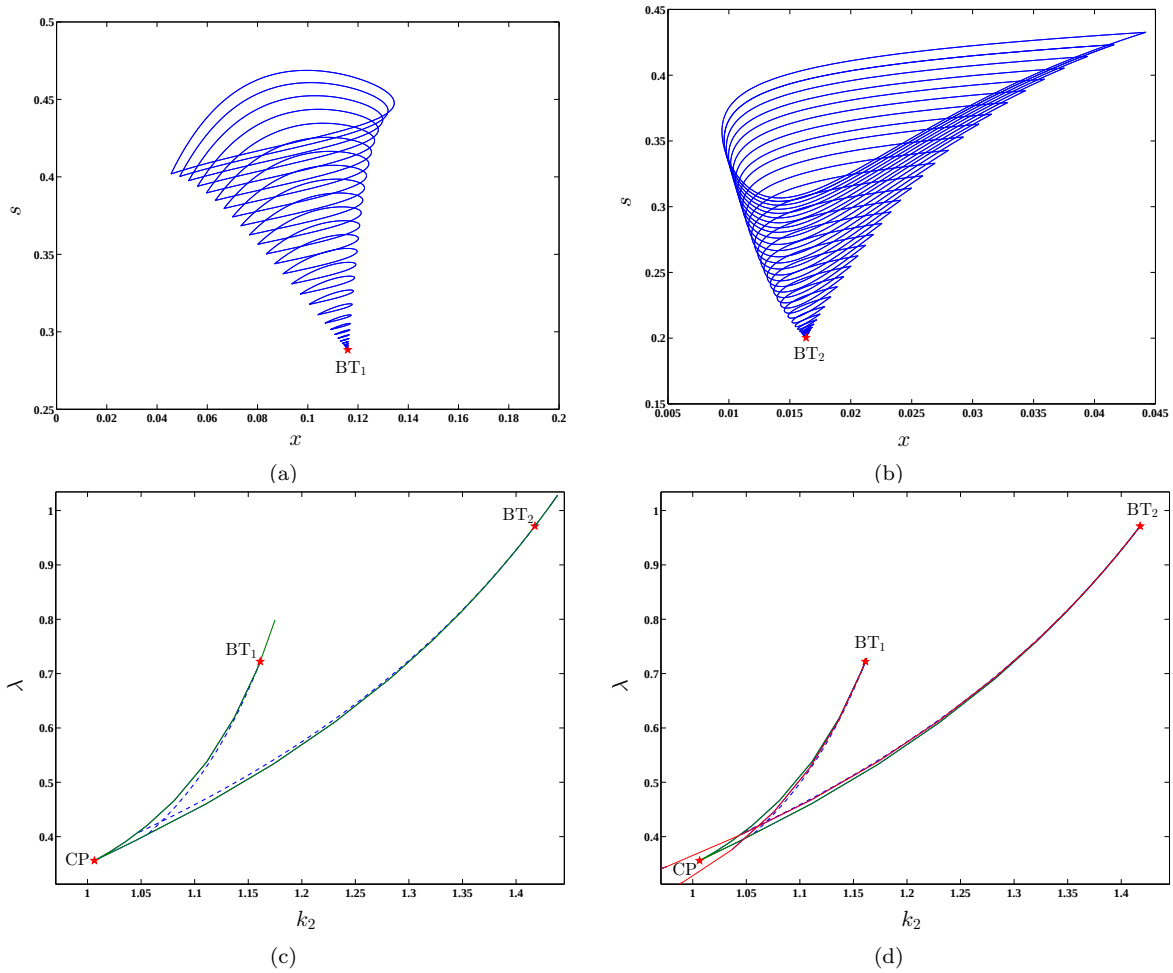


Figure 4: (a) and (b) Homoclinic orbits in (x, s)-space for the CO−oxidation model, (c) and (d) Homoclinic orbit in parameter space. The dashed blue curve is the homoclinic curve. The green curve is the LP curve and the red is the Hopf curve.

4.4 The extended Lorenz-84 model

This example is an extended version of the Lorenz-84 model. In this model we can find all codim 2 points of equilibria (i.e., BT, CP, GH, ZH and HH) [21, 24, 23]. Here, we discuss the switching to the homoclinic branches from the computed BT points.

The extension model of Lorenz-84 has the form

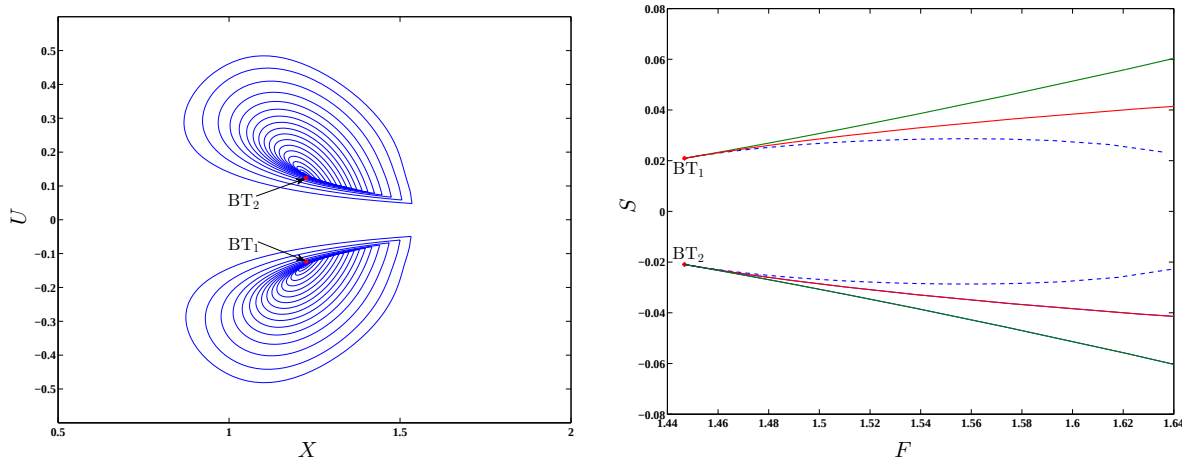
$$\begin{aligned}\dot{X} &= -Y^2 - Z^2 - \alpha X + \alpha F - \xi U^2,\\ \dot{Y} &= XY - \beta XZ - Y + G,\\ \dot{Z} &= \beta XY + XZ - Z,\\ \dot{U} &= -\delta U + \xi UX + S.\end{aligned} \tag{14}$$

In this system, X models the intensity of a baroclinic wave, Y and Z the sin and cos coefficients of the wave respectively, the variable U is added to study the influence of external parameters such as temperature. The parameters are fixed as follows $\alpha = 0.25$, $\beta = 1$, $G = 0.25$, $\delta = 1.04$, and $\xi = 0.987$, $F = 2.61$. Start the equilibria continuation with free $S = 0$ and with initial state variable values $X = 1.05$, $Y = -0.01$, $Z = 0.21$, $U = -0.5$. Two LP are detected. From any LP point compute the LP curve passing through it with (F, S) as a free system parameters. Two BT points are found (See Table 3). From BT_1 point start the

Table 3: Parameter, state and BT normal form coefficient (a, b) values at the bifurcation points in Fig.5.

Label	F	S	State variables	Normal form coefficients (a, b)
BT_1	1.45	0.02	$(1.23, -0.04, 0.20, -0.12)$	$(-0.21, -0.61)$
BT_2	1.45	-0.02	$(1.23, -0.04, 0.20, 0.12)$	$(0.21, 0.61)$

homoclinic continuation with (F, S) free, `kfactor` $= 100$, $\varepsilon = 4 \times 10^{-4}$, ntst=40, ncol=4, Adapt $= 1$, $(\varepsilon_0, \varepsilon_1)$ free homoclinic parameters. In the **Continuer** window set the **MaxStepSize** $= 0.2$ and then **Compute** - **Forward**. You should obtain the homoclinc orbits as in Fig.5. Notice that for BT_2 the same procedure with fixed $T = 14720.40$ and then **Compute** - **Forward** can be used.



(a) (b)

Figure 5: (a) Homoclinic orbits in $(X, U)-$space for the CO$-$oxidation model, (b) Homoclinic orbits in parameter space. The dashed blue curve is the homoclinic curve. The red is the Hopf curve and the green is the LP curve.

5 Conclusion and suggestions

We have described the MatCont initializer to start homoclinic orbits from a given BT point. The intializer is used to assign the initial cycle and homoclinic parameters so that the homoclinic continuation can start.

We suggest to allow ε_0 and ε_1 to vary as homoclinic parameters then start increase/decrease the amplitude value ε and for each ε use different values for `kfactor`. This works for most studied models. However, it does take some trial-and-error to set all parameters (including the continuation and corrector data) for the continuation.

Note that in each case both **Compute - Forward** and **Compute - Backward** should be tried.

In general, cases where a, b have a comparable size (in absolute value) are easier to handle than cases where they differ by a large factor.

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