Introducción a las Finanzas cuánticas

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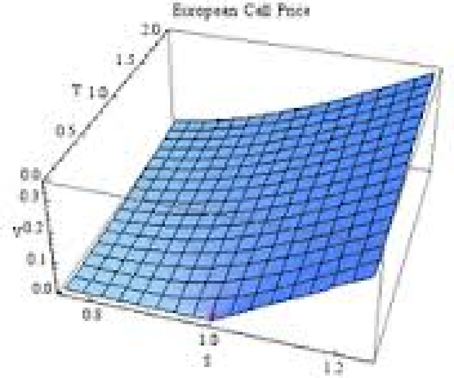
Casa abierta al tiempo

ORDEN DE LA PRESENTACIÓN

- Partícula libre
- Mecánica cuántica
- Black-Scholes
- Integral de trayectoria
- Perspectivas

Finanzas

- Black-Scholes
- $$\frac{\partial C(s,t)}{\partial t} = -\frac{\sigma^2}{2} s^2 \frac{\partial^2 C(s,t)}{\partial s^2} - rs\frac{\partial C(s,t)}{\partial s} + rC(s,t), \tag{1}$$



Partícula libre y Ecuación de Black-Scholes

- Partícula libre en una dimensión
- $$S = \int dt \frac{m}{2} \left(\frac{dx}{dt} \right)^2 . \qquad (2)$$
- Segunda ley de Newton

$$F = m\ddot{x} = 0. \qquad (3)$$

- Simetrías
- $t' = t + \beta, \qquad x' = x,$
- $t' = t, \quad x' = x + c,$
- $t' = t, \qquad x' = x + vt,$
- $t' = a^2 t \qquad x' = ax,$
- $t' = \frac{1}{\gamma t + 1}, \qquad x' = \frac{x}{\gamma t + 1}$

Partícula libre

- $$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad x' = \frac{ax + vt + c}{\gamma t + \delta}, \qquad a^2 = \alpha\delta - \beta\gamma \neq 0, \quad (4)$$

- $$S' = \int dt' \frac{m}{2}\left(\frac{dx'}{dt'}\right)^2 = \int dt \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{m}{2}\int dt \left(\frac{d\phi(x,t)}{dt}\right).$$

  donde

- $$\phi(x,t) = \frac{1}{a^2}\left(2avx + v^2 t - \frac{\gamma\,(ax + vt + c)^2}{\gamma t + \delta}\right). \qquad (5)$$

Cantidades conservadas

- $P = m\dot{x}$,
- $H = \frac{P^2}{2m}$,
- $G = tP - mx$,
- $K_1 = tH - \frac{1}{2}xP$,
- $K_2 = t^2H - txP + \frac{m}{2}x^2$

$$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

Algebra de Schrödinger

- $\{P, H\} = 0,$
- $\{P, K_1\} = \frac{1}{2}P,$
- $\{P, K_2\} = G,$
- $\{P, G\} = m,$
- $\{H, K_1\} = H,$
- $\{H, G\} = P,$
- $\{H, K_2\} = 2K_1,$
- $\{K_1, K_2\} = K_2,$
- $\{K_1, G\} = \frac{1}{2}G,$
- $\{K_2, G\} = 0$

Fundamentos de Mecánica Cuántica

- Principio de incertidumbre $\Delta x \Delta p \geq \frac{\hbar}{2}$, $\qquad p = m\dot{x}$
- cuantización $p \to \hat{p} = -i\hbar\frac{\partial}{\partial x}$
- Ecuación de Schrödinger
- $$i\hbar\frac{\partial\psi\,(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}. \tag{6}$$
- $P = \psi^*(x,t)\psi(x,t)dx$
- Heisenberg $\{A,B\} \to [\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

Integral de Trayectoria I

- Propagador

$$U(z,t;z_0,t_0)=\left(\frac{m}{2\pi i\hbar(t-t_0)}\right)^{\frac{1}{2}} e^{\frac{im}{2\hbar}\frac{(z-z_0)^2}{(t-t_0)}}, \tag{7}$$

- Kolmogorov-Chapman $t_0 < t_1 < t_2$

$$U(z_2,t_2;z_0,t_0)=\int_{-\infty}^{\infty} dz_1 U(z_2,t_2;z_1,t_1)U(z_1,t_1;z_0,t_0) \tag{8}$$

- Además, si $x_0 < x_1 < x_2 < x_3$ y $t_0 < t_1 < t_2 < t_3$,

$$U\left(x_3,t_3;x_0,t_0\right)=\int_{-\infty}^{\infty} dx_2 U\left(x_3,t_3;x_2,t_2\right) U\left(x_2,t_2;x_0,t_0\right)$$

$$=\int_{-\infty}^{\infty} dx_2 dx_1 U\left(x_3,t_3;x_2,t_2\right) U\left(x_2,t_2;x_1,t_1\right) U\left(x_1,t_1;x_0,t_0\right)$$

Integral de Trayectoria II

En general, si

$$x_0 < x_1 < x_2 \cdots < x_{N-1} < x_N = x_f$$

y

$$t_0 < t_1 < t_2 \cdots < t_{N-1} < t_N = t_f,$$

entonces se encuentra

$$U(x_f, t_f; x_0, t_0) = \int_{-\infty}^{\infty} dx_1 \cdots x_{N-1}$$
$$U(x_N, t_N; x_{N-1}, t_{N-1})\, U(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2}) \cdots U(z_1, t_1; x_0, t_0$$

es decir

$$U(z_f, t_f; x_0, t_0) = \int_{-\infty}^{\infty} \Pi_{l=1}^{N-1} dx_l \Pi_{n=0}^{N-1} U(z_{n+1}, t_{n+1}; x_n, t_n). \quad (9)$$

Integral de Trayectoria III

- $$\Pi_{n=0}^{N-1} K\left(z_{n+1}, t_{n+1}; x_n, t_n\right) = \left(\Pi_{n=0}^{N-1} \sqrt{\frac{m}{2\pi i\hbar\left(t_{n+1}-t_n\right)}}\right)$$
$$e^{\frac{im}{2\hbar}\sum_{n=0}^{N-1}\frac{\left(x_{n+1}-x_n\right)^2}{t_{n+1}-t_n}}$$

$$\Delta x_n = x_{n+1} - x_n, \qquad \Delta t_n = t_{n+1} - t_n. \tag{10}$$

$$\sum_{n=0}^{N-1} \frac{m}{2} \frac{\left(x_{n+1}-x_n\right)^2}{t_{n+1}-t_n} = \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{\Delta x_n}{\Delta t_n}\right)^2 \Delta t_n. \tag{11}$$

$\Delta t_n \to 0$

$$\frac{\Delta x_n}{\Delta t_n} \to \frac{dx}{dt}, \tag{12}$$

Integral de Trayectoria IV

$$\sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{\Delta x_n}{\Delta t_n} \right)^2 \Delta t_n \to \int_{t_0}^{t_f} \frac{m}{2} \left(\frac{dx}{dt} \right)^2 dt. \tag{13}$$

$$U\left(x_f, t_f; x_0, t_0\right) = \int Dx(t) e^{\frac{i}{\hbar} S(x(t))}, \tag{14}$$

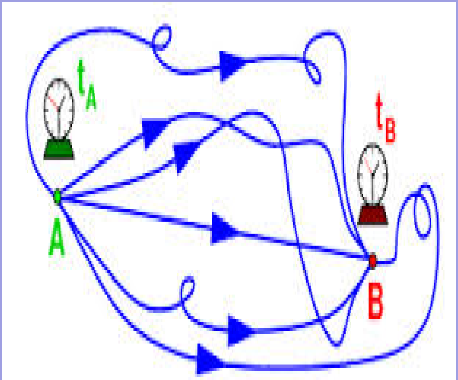
donde

$$S(x(t)) = \int_{t_0}^{t_f} dt \frac{m}{2} \left(\frac{dx}{dt} \right)^2, \tag{15}$$

y

$$Dx(t) = \lim_{\Delta t_n \to 0} \Pi_{l=1}^{N-1} dx_l \Pi_{n=0}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t_n}}. \tag{16}$$

Trayectorias



Ecuación de Schrödinger

- $$i\hbar\frac{\partial\psi\left(x,t\right)}{\partial t}=-\frac{\hbar^{2}}{2m}\frac{\partial^{2}\psi(x,t)}{\partial x^{2}}. \tag{17}$$

- $$t' = \frac{\alpha t+\beta}{\gamma t+\delta}, \qquad x' = \frac{ax+vt+c}{\gamma t+\delta}, \tag{18}$$
$$\psi'\left(x',t'\right) = \left(\sqrt{\gamma t+\delta}\right)e^{\frac{im}{2\hbar}\phi(x,t)}\psi(x,t), \tag{19}$$
$$\phi(x,t) = \frac{1}{a^{2}}\left(2avx+v^{2}t-\frac{\gamma\left(ax+vt+c\right)^{2}}{\gamma t+\delta}\right). \tag{20}$$

Operadores

- $$\begin{aligned} \hat{P} &= -i\hbar\frac{\partial}{\partial x}, && (21)\\ \hat{H} &= \frac{\hat{P}^2}{2m}, && (22)\\ \hat{G} &= t\hat{P} - mx, && (23)\\ \hat{K}_1 &= t\hat{H} - \frac{1}{4}\left(x\hat{P} + \hat{P}x\right), && (24)\\ \hat{K}_2 &= t^2\hat{H} - \frac{t}{2}\left(x\hat{P} + \hat{P}x\right) + \frac{m}{2}x^2, && (25) \end{aligned}$$

Algebra de operadores

- $\left[\hat{P}, \hat{H}\right] = 0,$
- $\left[\hat{P}, \hat{K}_1\right] = \frac{i\hbar}{2}\hat{P},$
- $\left[\hat{P}, \hat{K}_2\right] = i\hbar\hat{G},$
- $\left[\hat{P}, \hat{G}\right] = i\hbar m,$
- $\left[\hat{H}, \hat{K}_1\right] = i\hbar\hat{H},$
- $\left[\hat{H}, \hat{G}\right] = i\hbar\hat{P},$
- $\left[\hat{H}, \hat{K}_2\right] = 2i\hbar\hat{K}_1,$
- $\left[\hat{K}_1, \hat{K}_2\right] = i\hbar\hat{K}_2,$
- $\left[\hat{K}_1, \hat{G}\right] = \frac{i\hbar}{2}\hat{G},$
- $\left[\hat{K}_2, \hat{G}\right] = 0.$
- Niederer, Hagen in 1972, S. Lie, 1882. Ecuación de Fick.

Black-Scholes

- $$\frac{\partial C(s,t)}{\partial t} = -\frac{\sigma^2}{2}s^2\frac{\partial^2 C(s,t)}{\partial s^2} - rs\frac{\partial C(s,t)}{\partial s} + rC(s,t), \tag{26}$$
- $s = e^x$
- $\frac{\partial C(x,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 C(x,t)}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial C(x,t)}{\partial x} + rC(x,t).$

Black-Scholes

- $C(x,t) = e^{\left[\frac{1}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right)x+\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2 t\right]}\psi(x,t)$
- $$\frac{\partial\psi(x,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}. \tag{27}$$
- Simetría Conforme

$$t' = \frac{\alpha t+\beta}{\gamma t+\delta}, \qquad x' = \frac{ax+vt+c}{\gamma t+\delta}, \tag{28}$$

$$\psi'\left(x',t'\right) = \left(\sqrt{\gamma t+\delta}\right) e^{\frac{1}{2\sigma^2}\phi(x,t)}\psi(x,t), \tag{29}$$

$$\phi(x,t) = \frac{1}{a^2}\left(2avx+v^2 t-\frac{\gamma\left(ax+vt+c\right)^2}{\gamma t+\delta}\right). \tag{30}$$

Grupo de Schrödinger y Black-Scholes

- $$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad s' = e^{\left(\frac{vt+c}{\gamma t+\delta}\right)} s^{\left(\frac{a}{\gamma t+\delta}\right)}. \tag{31}$$
- Precio
- $C'(s', t') =$

$$\left(\sqrt{\gamma t + \delta}\right) \left(s^{\frac{-2a^2\gamma\left(\frac{\sigma^2}{2}-r\right)t+2a(v\delta-\gamma c)+2a^2(a-\delta)\left(\frac{\sigma^2}{2}-r\right)-\gamma a^2(\ln s)}{2a^2\sigma^2(\gamma t+\delta)}} \right)$$

$$\mathrm{e}^{\frac{-\gamma a^2\left(\frac{\sigma^2}{2}+r\right)^2 t^2+\left(a^2\left(\frac{\sigma^2}{2}+r\right)^2(\alpha-\delta)+2a^2 v\left(\frac{\sigma^2}{2}-r\right)+v(v\delta-2\gamma c)\right)t}{2\sigma^2 a^2(\gamma t+\delta)}}$$

$$\mathrm{e}^{\frac{a^2\beta\left(\frac{\sigma^2}{2}+r\right)^2+2a^2\left(\frac{\sigma^2}{2}-r\right)c-\gamma c^2}{2\sigma^2 a^2(\gamma t+\delta)}} C(s,t).$$

Hamiltoniano de Black-Scholes

- $$\hat{\mathbf{H}} = -\frac{\sigma^2}{2} s^2 \frac{\partial^2}{\partial s^2} - rs\frac{\partial}{\partial s} + r, \tag{32}$$

- $$\frac{\partial C(s,t)}{\partial t} = \hat{\mathbf{H}} C(s,t). \tag{33}$$

- $$\hat{\Pi} = -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right), \tag{34}$$

- $$\hat{\mathbf{H}} = \frac{\sigma^2}{2}\hat{\Pi}^2 + \frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2} + r\right)^2. \tag{35}$$

Grupo de Schrödinger y Black-Scholes

$$\hat{\Pi} = -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2} - r\right), \tag{36}$$

$$\hat{\mathbf{H}}_0 = \frac{\sigma^2}{2}\hat{\Pi}^2, \tag{37}$$

$$\hat{\mathbf{G}} = t\hat{\Pi} - \frac{t}{\sigma^2}\ln s, \tag{38}$$

$$\hat{\mathbf{K}}_1 = t\hat{\mathbf{H}}_0 - \frac{1}{4}\left(\ln s\hat{\Pi} + \hat{\Pi}\ln s\right), \tag{39}$$

$$\hat{\mathbf{K}}_2 = t^2\hat{\mathbf{H}}_0 - \frac{t}{2}\left(\ln s\hat{\Pi} + \hat{\Pi}\ln s\right) + \frac{1}{2\sigma^2}(\ln s)^2 \tag{40}$$

Grupo de Schrödinger y Black-Scholes

- $[\ln s, \hat{\Pi}] = i,$
- $\left[\hat{\Pi}, \hat{\mathbf{H}}\right] = 0,$
- $\left[\hat{\Pi}, \hat{\mathbf{K}}_1\right] = \frac{i}{2}\hat{\Pi},$
- $\left[\hat{\Pi}, \hat{\mathbf{K}}_2\right] = i\hat{\mathbf{G}},$
- $\left[\hat{\Pi}, \hat{\mathbf{G}}\right] = \frac{i}{\sigma^2},$
- $\left[\hat{\mathbf{H}}, \hat{\mathbf{K}}_1\right] = i\hat{\mathbf{H}}_0$
- $\left[\hat{\mathbf{H}}, \hat{\mathbf{G}}\right] = i\hat{\Pi},$
- $\left[\hat{\mathbf{H}}, \hat{\mathbf{K}}_2\right] = 2i\hat{\mathbf{K}}_1, \qquad \left[\hat{\mathbf{K}}_1, \hat{\mathbf{K}}_2\right] = i\hat{\mathbf{K}}_2,$
- $\left[\hat{\mathbf{K}}_1, \hat{\mathbf{G}}\right] = \frac{i}{2}\hat{\mathbf{G}}, \qquad \left[\hat{\mathbf{K}}_2, \hat{\mathbf{G}}\right] = 0.$

Integral de Trayectoria I

- Solución

$$C(x,t) = \int_{-\infty}^{\infty} dx' G\left(x,t;x',t'\right) C(x',t'). \tag{41}$$

$$C(S,t) = SN(d_+) - Ke^{-r(T-t)}N(d_-). \tag{42}$$

- Propagador

$$G\left(x,t;x',t'\right) = \frac{e^{-r(t-t')}}{\sqrt{2\pi\sigma^2(t-t')}} e^{-\frac{\left((x-x')-(t-t')\left(\frac{\sigma^2}{2}-r\right)\right)^2}{2\sigma^2(t-t')}}, \tag{43}$$

- Kolmogorov-Chaptman

$$G(x_2,t_2;x_0,t_0) = \int_{-\infty}^{\infty} dx_1 G(x_2,t_2;x_1,t_1)G(x_1,t_1;x_0,t_0) \tag{44}$$

Integral de Trayectoria

Si $x_0 < x_1 < x_2 \cdots < x_{N-1} < x_N = x_f, t_0 < t_1 < t_2 \cdots < t_{N-1} < t_N = t_f$,

$$G(x_f, t_f; x_0, t_0) = \int_{-\infty}^{\infty} \Pi_{l=1}^{N-1} dx_l \Pi_{n=0}^{N-1} G(x_{n+1}, t_{n+1}; x_n, t_n) \,. \quad (45)$$

$$\Pi_{n=0}^{N-1} G(z_{n+1}, t_{n+1}; x_n, t_n) = \left(\Pi_{n=0}^{N-1} \sqrt{\frac{1}{2\pi\sigma^2 (t_{n+1} - t_n)}} \right)$$

$$e^{-\sum_{n=0}^{N-1} r(t_{n+1}-t_n)} e^{\frac{-1}{2\sigma^2} \sum_{n=0}^{N-1} \frac{\left((x_{n+1}-x_n)-(t_{n+1}-t_n)\left(\frac{\sigma^2}{2}-r\right)\right)^2}{t_{n+1}-t_n}}$$

Integral de Trayectoria

- $\Delta t_n = t_{n+1} - t_n, \Delta x_n = x_{n+1} - x_n$

$$\Pi_{n=0}^{N-1} G\left(z_{n+1}, t_{n+1}; x_n, t_n\right) = \left(\Pi_{n=0}^{N-1} \sqrt{\frac{1}{2\pi\sigma^2 \Delta t_n}}\right)$$

$$e^{-\sum_{n=0}^{N-1} \Delta t_n \left[r + \frac{1}{2\sigma^2}\left[\frac{\Delta x_n}{\Delta t_n} - \left(\frac{\sigma^2}{2} - r\right)\right]^2\right]}$$

Entonces $\Delta t_n \to 0$

$$\sum_{n=0}^{N-1} \Delta t_n \left[r + \frac{1}{2\sigma^2}\left[\frac{\Delta x_n}{\Delta t_n} - \left(\frac{\sigma^2}{2} - r\right)\right]^2\right] \to \tag{46}$$

$$\int_{t_0}^{t_f} dt \left[r + \frac{1}{2\sigma^2}\left[\frac{dx}{dt} - \left(\frac{\sigma^2}{2} - r\right)\right]^2\right] \tag{47}$$

Integral de Trayectoria

$$G\left(x_f, t_f; x_0, t_0\right) = \int Dx(t) e^{-S_{BS}(x(t))}, \quad (48)$$

donde

$$S_{BS}(x(t)) = \int_{t_0}^{t_f} dt \left(r + \frac{1}{2\sigma^2} \left[\frac{dx}{dt} - \left(\frac{\sigma^2}{2} - r \right) \right]^2 \right) \quad (49)$$

y

$$Dx(t) = \lim_{\Delta t_n \to 0} \left(\Pi_{l=1}^{N-1} dx_l \Pi_{n=0}^{N-1} \sqrt{\frac{1}{2\pi\sigma^2 \Delta t_n}} \right). \quad (50)$$