Sufficient conditions for isomorphisms between function algebras, Thomas Tonev, The University of Montana, Missoula

Let $T\colon A \to B$ be a surjective (not necessarily linear) map between two function algebras on locally compact Hausdorff spaces X and Y with Choquet boundaries $\delta A \subset X$ and $\delta B \subset Y$. If $\|Tf\,Tg\| = \|fg\|$ for all $f, g \in A$, then there is a homeomorphism $\psi\colon \delta B \to \delta A$ so that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. If, more generally, $\sigma_\pi(Tf\,Tg) \subset \sigma_\pi(fg)$ for all $f, g \in A$ (where $\sigma_\pi(f)$ is the peripheral spectrum of f), we show that $(Tf)(y) = \alpha(y)\, f(\psi(y))$ for some $\alpha \in C(\delta B)$ with $\alpha^2 = 1$, i.e. T is a weighted composition operator on δB. This is true also if $\sigma_\pi(Tf\,Tg) \cap \sigma_\pi(fg) \neq \emptyset$ for all $f, g \in A$ and T preserves singleton peripheral spectra of algebra elements. In the case of metric spaces X the condition $\sigma_\pi(Tf\,Tg) \cap \sigma_\pi(fg) \neq \emptyset$ alone suffices for T to be a weighted composition operator. If, in addition $dist(\sigma_\pi(Tf), \sigma_\pi(f)) < 2$ for all $f \in A$, then in all cases $\alpha = 1$, i.e. $(Tf)(y) = f(\psi(y))$, therefore, T is a composition operator, and consequently, an isometric algebra isomorphism. (Jointly with J. Johnson, PhD student)

More generally, if $\|Tf\,Tg\| = \|fg\|$ and there is an $0 \leq \varepsilon < 2/3$, so that $\sigma_\pi(Tf\,Tg)$ is contained in an $(\varepsilon\,\|fg\|)$-neighborhood of $\sigma_\pi(fg)$ for all $f \in A$ and all $g \in A$ with $\|g\| = 1$, then there is an $\alpha \in C(\delta B)$ with $\alpha^2 = 1$ and a homeomorphism $\psi\colon \delta B \to \delta A$ so that $|(Tf)(y) - \alpha(y)\, f(\psi(y))| \leq 2\varepsilon\, |f(\psi(y)|$ for each $f \in A$ and every $y \in \delta B$, i.e. T is an almost weighted composition operator on δB. Moreover, if there are $0 \leq \varepsilon < 1$, $0 \leq \eta < 1$, so that $dist(\sigma_\pi(Tf\,Tg), \sigma_\pi(fg)) \leq \varepsilon\,\|fg\|$ and $\sigma_\pi(Tf)$ is contained in an η-neighborhood of $\sigma_\pi(f)$ for all $f \in A$ and all $g \in A$ with $\|g\| = 1$, then $|(Tf)(y) - f(\psi(y))| \leq (\varepsilon + \eta)\, |f(\psi(y))|$ for each $y \in \delta B$ and every $f \in A$, i.e. T is an almost algebraic isomorphism. (To appear in the PAMS)