A Continuity Correction under Jump-Diffusion Models with Applications in Finance

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Abstract. We show some discrete path-dependent options which can be approximately priced by using their continuous counterparts' pricing formulas with a simple continuity correction under jump diffusion models. Based on exponential martingale and functional central limit theorem with appropriate scaling of the underlying process, we first apply Siegmund's corrected diffusion method to get an approximation of the discrete barrier option. Then, we extend our result to get a continuity correction of the discrete lookback option via a key observation, to which a lookback option can be regarded as a kind of *moving barrier* options. An interesting phenomenon of the discrete option pricing is that the correction term depends only on σ (the volatility of Brownian motion part) and $\beta \approx 0.5826$ (the expected overshoot), and does not depend on the jump part of the underlying process. Numerical results also show that the performance is very accurate as well.

Keywords. continuity corrections, jump diffusion models, Laplace transforms, path-dependent options, renewal theorem.

1 Introduction

In this paper we investigate the problem of connecting the discrete- and continuous-version path-dependent options under jump diffusion models. Due to the jump part, when a jump diffusion process crosses a boundary level, it may incur an "overshoot" phenomenon over the boundary even under continuously monitoring. The overshoot caused by either jump effect or discretization effect (or both) consequently becomes difficult to recognize when we would like to price the discrete path-dependent options. However, based on exponential martingale and functional central limit theorem with appropriate scaling of the underlying process, the (first-order) corrected diffusion approximation only corrects the overshoot caused by the discretization on diffusion part. That is, the correction term depends only on σ, the volatility of Brownian motion part, and β, the expected overshoot of sum of i.i.d. standard normal random variables, and does not depend on the jump part of the underlying process.

It is important to stress that this correction issue is eloquent and meaningful only when the continuous option pricing formulas can be expressed explicitly in closed-form, since the purpose of the present paper is to price discrete path-dependent options by using their continuous counterparts' pricing formulas with a simple continuity correction. Hence, we mainly focus on two special cases of the jump-diffusion processes in the sequel, which including the double exponential jump-diffusion model [cf. Kou (2002)] and the spectrally one-sided jump-diffusion model [cf. Rogers (2000)]. A unique feature of these models is the memoryless property which is necessary to obtain the analytical solutions for option pricing. This special property explains why the closed-form solutions for various option-pricing problems, including barrier and lookback options, are feasible under these models while it seems impossible for many other models, including the normal jump-diffusion model, cf. Merton (1976).

Moreover, by making use of a key observation, we represent a floating strike lookback call option as a kind of down-and-in lookback-style moving barrier put option. This devise enables us to evaluate the lookback options via the results of barrier options.

2 Model settings

Assume the price of the underlying asset $S(t), t \geq 0$, is given as the solution to the stochastic differential equation (SDE):

$$dS(t) = (r - \lambda\delta)S(t)dt + \sigma S(t)dW(t) + S(t-)d(\sum_{j=1}^{N(t)}(J_j - 1)), \tag{1}$$

under the risk-neutral probability measure P^* [cf. Kou (2002)], where $W(t)$ is the Wiener process, $N(t)$ is a Poisson process with intensity λ, and $\{J_j\}$ is a sequence of independent and identically distributed (i.i.d.) nonnegative random variables, such that $\Upsilon = \log J$ denotes the jump size with the density $f_\Upsilon(y)$. In equation (1), r denotes the constant of risk free rate, σ is the volatility and $\delta = \mathrm{E}^*[J_1] - 1$. In this model all sources of randomness, $W(t)$, $N(t)$, and Js, are assumed to be independent. The solution of equation (1) is

$$\begin{aligned} S(t) &= S(0)e^{rt}\exp\{\sigma W(t) - \frac{1}{2}\sigma^2 t - \lambda\delta t\} \cdot \Pi_{j=1}^{N(t)} J_j \\ &= S(0)\exp\{\sigma W(t) + (\mu + \lambda\xi)t + M(t)\} \equiv S(0)\exp\{X(t)\},^{1} \end{aligned} \tag{2}$$

where $\mu = r - \frac{1}{2}\sigma^2 - \lambda\delta$, $\xi = \mathrm{E}^*[\Upsilon_1]$, such that $M(t) = \sum_{j=1}^{N(t)} \Upsilon_j - \lambda\xi t$ is the compensated compound Poisson process and $X(0) = 0$.

Now, suppose that the underlying asset price is monitored only at time $i\Delta t, i = 1, 2, \ldots, m$, where $\Delta t = T/m$, $T > 0$ is fixed, and m is the frequency of monitoring. Then, under the risk-neutral probability measure P^*, at the n-th monitoring point, time $n\Delta t$, the discrete version of our asset price is given by, n=1, ..., m,

$$\begin{aligned} S_n &= S(0)\exp\left\{\sigma\sqrt{\Delta t}\sum_{i=1}^{n} Z_i + (\mu + \lambda\xi)n\Delta t + \sum_{i=1}^{n}(\sum_{j=1}^{N_i}\Upsilon_j - \lambda\xi\Delta t)\right\} \\ &= S(0)\exp\left\{\sum_{i=1}^{n}\left(\sigma\sqrt{\Delta t}\cdot Z_i + (\mu + \lambda\xi)\Delta t + M_i\right)\right\} \equiv S(0)\exp\{X_n\}, \end{aligned} \tag{3}$$

where $Z_i \overset{i.i.d.}{\sim} N(0,1)$, $N_i \overset{i.i.d.}{\sim} Poisson(\lambda\Delta t)$, and $M_i = \sum_{j=1}^{N_i}\Upsilon_j - \lambda\xi\Delta t$. Note that $\{Z_i\}$ and $\{N_i\}$ are independent, and $X_0 \equiv 0$.

In addition, suppose that $\theta \in I \in \mathbb{R}$. Denoted by $\mathrm{M}_\Upsilon(\theta)$ the moment generating function of jump size Υ. Thanks to the celebrated *Lévy-Khintchine formula for Lévy processes*, the moment generating function of $X(t)$ can be obtained as $\mathrm{E}[e^{\theta X_t}] = e^{G(\theta)\cdot t}$, $\forall t > 0$; similarly, $\mathrm{E}[e^{\theta X_n}] = e^{G(\theta)\cdot n\Delta t}$, $\forall n > 0$, where $G(\cdot)$ is defined as

$$G(\theta) = \frac{1}{2}\sigma^2\theta^2 + \mu\theta + \lambda[\mathrm{M}_\Upsilon(\theta) - 1].$$

We assume $\mathrm{M}_\Upsilon(\theta)$ exist for any $\theta \in I \subset \mathbb{R}$ and is differentiable at $\theta = 0$, particularly, its first and second derivatives at zero are exist, which implies that $\mathrm{E}^*[\Upsilon_1]$ and $\mathrm{E}^*[\Upsilon_1^2]$ exist.

In the present paper, we concentrate on two special cases of jump-diffusion models. These two have the lack of memory property in common. We separate the two cases from behavior of the jump.

- First, we consider the spectrally one-sided jump-diffusion model, in which $\mathrm{M}_\Upsilon(\theta)$ exists for all $\theta \in \mathbb{R}$ and the underlying asset jumps in opposite direction to the barrier (or extreme level) such that there will no overshoot problems occur in the continuous paths when crossing the boundary. Hence, the effect of overshoot is only caused by discretization. In this case, the continuity correction problem reduce justly to that in the Black-Scholes model and therefore we will get the consistent results with Broadie et al. (1999).

[1] $X(t)$ can be regarded as a special case of Lévy processes, see Kyprianou (2006) for reference.

- The second one, a double exponential jump-diffusion model(DEJM) proposed by Kou (2002) will be focused on, which is a special case of two-sided jump-diffusion model. We use this model due to the fact that the model can lead to analytical solutions for lookback and barrier options thanks to the memoryless property of the double exponential distribution. In this model, we assume the jump size, $\Upsilon = \log J$, has an asymmetric double exponential distribution with the density $f_\Upsilon(y) = p \cdot \eta_1 e^{-\eta_1 y} I_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} I_{\{y<0\}}, \eta_1 > 1,\ \eta_2 > 0$, where $p, q \geq 0, p + q = 1$. Here I denotes the indicator function and $\delta = \mathrm{E}^*[J_1] - 1 = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2+1} - 1$. Note that the condition $\eta_1 > 1$ is imposed to ensure that the asset price $S(t)$ has finite expectation. Suppose that $\theta \in (-\eta_2, \eta_1)$, the function $G(\theta)$ is defined as

$$G(\theta) = \mu\theta + \frac{1}{2}\theta^2\sigma^2 + \lambda(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + \theta} - 1).$$

In the sequel, the results derived in the present paper are all under the two special cases of jump-diffusion models.

For any process $X(t)$, let τ, $\tilde{\tau}$ be the first-passage (stopping) times

$$\tau(b) = \tau(b, X) := \inf\{t > 0 : X(t) \geq b\}$$
$$\tilde{\tau}(b) := \tilde{\tau}(b, X) = \inf\{t > 0 : X(t) \leq b\},$$

where b is a fixed barrier. The discrete versions are denoted by

$$\tau_m(b) = \tau_m(b, X) := \inf\{n \geq 1 : X_n \geq b\}$$
$$\tilde{\tau}_m(b) = \tilde{\tau}_m(b, X) := \inf\{n \neq 1 : X_n \leq b\}.$$

Set the minima of X between the time interval as $\bar{M} = \min_{u\in[t,T]} X(u)$ and $\bar{M}_m = \min_{k\leq n\leq m} X_n$. Given a fixed $0 \leq t \leq T$, denoted by S_+ and S_- the *predetermined max* and *min*, respectively, such that

$$S_+ = S_+(t) = \max_{u\in[0,t]} S(u) = \max_{0\leq n\leq k} S_n;$$
$$S_- = S_-(t) = \min_{u\in[0,t]} S(u) = \min_{0\leq n\leq k} S_n,$$

where the discrete monitoring point denoted by $k = 1, 2, 3, ..., m$ such that $k\Delta t = t$. Note that when time t is given, we regard $S_\pm$ as a parameter here.

3 The main results

3.1 Barrier option pricing

From standard option pricing theorem, the price of a continuous barrier option will be the expectation, taken with respect to the risk-neutral probability P^*, of the discounted (with the discount factor being e^{-rT}, in which T is the time to maturity) payoff of the option. The price of a continuous up-and-in put (UIP) option is given by

$$V(H) = \mathrm{E}^*[e^{-rT}(K - S(T))^+ I_{\{\tau(H,S)\leq T\}}],$$

where $H > S(0)$ is the barrier, and $K \leq H$ is the strike price. Now, let $a = \log(K/S(0))$ and $b = \log(H/S(0))$. Then the option price of UIP can be further expressed as:

$$V(H) = e^{-rT} \int_{-\infty}^{a} P^*(X(T) \leq y, \tau(b) \leq T) S(0) e^y dy. \tag{4}$$

By analogy, we can also apply risk-neutral pricing theory to get the discrete UIP option price,

$$V_m(H) = e^{-rT} \int_{-\infty}^{a} P^*(X(T) \leq y, \tau_m(b)\Delta t \leq T) S(0) e^y dy. \tag{5}$$

The following theorem makes a connection between the prices of discrete- and continuous-time barrier options under Jump-Diffusion Models. From mathematical point of view, such convergence in probability indicates that the jump part does not affect the correction term which depends only on σ (the volatility of Brownian motion part) and $\beta \approx 0.5826$ (the expected overshoot)

Theorem 1. *Let $V(H)$ be the price of a continuous barrier option with the barrier H, and $V_m(H)$ be the price of an otherwise identical barrier option with m monitoring points. Then for any types of up $(H > S(0))$ discrete barrier options with, we have the approximation*

$$V_m(H) = V(He^{\beta\sigma\sqrt{T/m}}) + o(1/\sqrt{m}); \tag{6}$$

and for any types of down $(H < S(0))$ options, we hence have

$$V_m(H) = V(He^{-\beta\sigma\sqrt{T/m}}) + o(1/\sqrt{m}), \tag{7}$$

where $\beta = -\zeta(\frac{1}{2})/\sqrt{2\pi} \approx 0.5826$, with $\zeta(\cdot)$ being the Riemann zeta function.

3.2 Lookback option pricing

In this subsection, we will rewrite the lookback option as a moving barrier option. To start with this issue, let T be the maturity date, and denoted by K the fixed strike price. The arbitrage-free price of a continuous floating strike lookback call (LBC) option at an arbitrary time $0 < t < T$ is given by

$$\begin{aligned} V = V(S_-) &= e^{-r(T-t)}\mathrm{E}^*[S(T) - \min_{u\in[0,T]} S(u)|\mathcal{F}_t] \\ &= S(t) - e^{-r(T-t)}S_- + e^{-r(T-t)}\,\mathrm{E}^*[(S_- - S(t)e^{\bar{M}})^+|\mathcal{F}_t]. \end{aligned} \tag{8}$$

Denote $d = \log(S_-/S(t))$ as a constant for each fixed t. Then by making use of Markovian property of $S(t)$, integration by parts, and the fact that for $c \in \mathbb{R}^+$, $\{S(t)e^{\bar{M}} \leq c\} = \{\tilde{\tau}(c, S) \leq T\}$, we have the price of continuous floating LBC,

$$V(S_-) = S(t) - e^{-r(T-t)}S_- + e^{-r(T-t)} \int_{-\infty}^{d} S(t)e^x P^*(\tilde{\tau}(x) \leq T)dx. \tag{9}$$

Remark 1. Focus on the term $e^{-r(T-t)}\mathrm{E}^*[(S_- - S(t)e^{\bar{M}})^+|\mathcal{F}_t]$ in (8), we transform the discounted value of a lookback option to a kind of down-and-in barrier put option with barrier S_- and strike price S_-. Since S_- is a moving constant depending on the process of the asset price during $[0, t]$, we can regard it as a special moving barrier option [2]. The underlying asset of the moving barrier option, however, is replaced by its minima observation in $[t, T]$, therefore, we call it a *lookback-style moving barrier option*.

This term also can be used to price (European) fixed strike lookback put options with time dependent strike price S_-. That is, it connects the fixed strike LBP and the floating strike LBC options under jump diffusion models.

By analogy, the price of a discrete floating strike LBC option at the k^{th}, $1 \leq k \leq m$, monitoring point of time is given by

$$V_m(S_-) = S_k - e^{-r(m-k)\Delta t}S_- + e^{-r(m-k)\Delta t} \int_{-\infty}^{d} S_k e^x P^*(\tilde{\tau}_m(x) \leq m)dx. \tag{10}$$

Then, the following theorem makes a continuity correction for the discrete lookback options price from its continuous price counterpart.

[2] The barrier option with time dependent barrier value is called a moving barrier option, a similar idea can be found in Davydov and Linetsky (2001).

Theorem 2. *Let V be the price of a continuous floating strike lookback option, and V_m be the price of an otherwise identical discrete counterpart with m fixed monitoring points. Then, the price of a discrete lookback put (LBP) at the k^{th} time point and the price of a continuous LBP at time $t = k\Delta t$ satisfy*

$$V_m(S_+) = \left[e^{-\beta\sigma\sqrt{\Delta t}} V(S_+ e^{\beta\sigma\sqrt{\Delta t}}) + (e^{-\beta\sigma\sqrt{\Delta t}} - 1)S_t\right] + o(\frac{1}{\sqrt{m}}). \tag{11}$$

The approximation for the lookback call (LBC) options is

$$V_m(S_-) = -\left[e^{\beta\sigma\sqrt{\Delta t}} V(S_- e^{-\beta\sigma\sqrt{\Delta t}}) + (e^{\beta\sigma\sqrt{\Delta t}} - 1)S_t\right] + o(\frac{1}{\sqrt{m}}). \tag{12}$$

Here the constant $\beta \approx 0.5826$ is defined as Theorem 1.

3.3 Proofs of Theorem 1 and Theorem 2

Note that the key point in the proofs is to investigate the difference between equations (4) and (5) or between equations (9) and (10). That is, to study the joint distributions of $(X(T), \tau(b, X))$ and $(X(T), \tau_m(b, X)\Delta t)$ together with the distributions of $\tau(x, X)$ and $\tau_m(x, X)\Delta t$. Thus, we devote to prove the following theorem

Theorem 3. *Under the conditions of Theorem 2.1, we have for any fixed level b, as $m \to \infty$, then*

$$P^*(\tau_m(b, X)\Delta t \le T) = P^*(\tau(b + \beta\sigma\sqrt{T/m}, X) \le T) + o(\frac{1}{\sqrt{m}}). \tag{13}$$

Moreover,

$$P^*(\tau_m(x, X)\Delta t \le T) = P^*(\tau(x + \beta\sigma\sqrt{T/m}, X) \le T) + o(\frac{1}{\sqrt{m}}) \tag{14}$$

holds for all $x \in [d, \infty)$, where d is a constant.

Based on exponential martingale and functional central limit theorem with appropriate scaling of the underlying process, we can accomplish the proofs by applying the uniform renewal theorem.

4 Numerical results

In this section, we present numerical results to indicate the accuracy of the corrected diffusion approximation for discrete barrier and lookback options under DEJM. Examples only include a discrete UIP barrier option in equation (6), and a floating strike LBP option in equation (11), since the other options can be done similarly. The tables show that the continuous barrier and lookback option prices can differ transparently from the discrete barrier and lookback option prices by economically significant amounts. Note that the accuracy of the corrected diffusion approximation under spectrally one-sided jump-diffusion model is even better than the accuracy under DEJM.

According to Table 1, AE stands for the absolute error between corrected continuous prices and discrete prices, while RE stands for the relative error calculated by $AE/V_m(H)$. Hence, it is clear to see that the resulting approximation is accurate, especially for H farther from K which is indicated in renewal theorem. With this approximation, we can get the prices of the discrete options more efficiently if there is a convenient way to calculate the values of the continuous options.

Since the LBP option can be regarded as the moving barrier UIC option, the numerical results of the LBP are similar to those of UIC options. The difference is that UIC considers different levels of barrier H; while LBP options studies different levels of predetermined maximum S_+. Consequently, the proposed approximation is more accurate for larger difference between the predetermined maximum and the current asset price.

Table 1 Up-and-in put option price results under DEJM, $m = 50$, varying H

H	True $V_m(H)$	Continuous barrier $V(H)$	Corrected continuous barrier $V(He^{\beta\sigma\sqrt{T/m}})$	AE	RE (%)
92	2.059	2.613	2.061	0.002	0.1
93	1.606	2.072	1.607	0.001	0.1
94	1.233	1.620	1.234	0.001	0.1
95	0.934	1.249	0.935	0.001	0.1
96	0.698	0.950	0.698	0.000	0.0
97	0.513	0.712	0.513	0.000	0.0
98	0.372	0.526	0.372	0.000	0.0
99	0.266	0.383	0.266	0.000	0.0
100	0.188	0.275	0.188	0.000	0.0
101	0.131	0.195	0.131	0.000	0.0
102	0.090	0.137	0.090	0.000	0.0
103	0.061	0.095	0.061	0.000	0.0
104	0.041	0.065	0.041	0.000	0.0

Option parameters: $S(0) = 90$, $K = 90$, $r = 0.1$, $\sigma = 0.3$, $\lambda = 1$, $p = 0.5$, $\eta_1 = \eta_2 = 30$, and $T = 0.2$. $\beta \approx 0.5826$. Assuming 250 trading days per year, $m = 50$ monitoring points roughly corresponds to daily monitoring of the barrier.

Table 2 Lookback put option price results under DEJM, $m = 50$, varying S_+

S_+	True $V_m(S_+)$	Continuous lookback $V(S_+)$	Corrected continuous lookback approximation	AE	RE (%)
105	10.489	10.137	10.467	0.022	0.2
110	12.608	11.978	12.588	0.020	0.2
115	15.663	14.798	15.646	0.017	0.1
120	19.397	18.346	19.386	0.011	0.1
125	23.592	22.395	23.585	0.007	0.0
130	28.076	26.767	28.072	0.004	0.0
135	32.733	31.335	32.731	0.002	0.0
140	37.491	36.018	37.491	0.000	0.0
145	42.307	40.767	42.307	0.000	0.0

Option parameters: $S(t) = 100$, $r = 0.1$, $\sigma = 0.3$, $\lambda = 1$, $p = 0.5$, $\eta_1 = \eta_2 = 10$, and $T = 0.2$. $\beta \approx 0.5826$.

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