Multi-stage procedures for estimation in a random one way model

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Abstract. Hamdy et al. (1992) introduced various multi-stage procedures to construct fixed-width confidence intervals for the mean of the random one-way model. Using a more general set, in this paper, multi-stage procedures like, two-stage and three-stage procedure are proposed for the minimum risk point estimation of the mean of a random one way model taking loss function to be general absolute with general cost of sampling. Second-order asymptotics for the proposed estimation procedures are also obtained. The model can be adapted to the process of sub-sampling with equal sizes where variation among the primary units as well as the secondary units is unknown.
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1 Formulation of the problem

Consider the model $Y_{ij} = \mu + \tau_i + e_{ij}$, i=1,2,…,n; j=1,2,…, r_1 (1)

where τ_i ~NID(0, σ_t^2) and e_{ij} ~NID(0, σ_e^2). $\Omega = \{\mu \in \Re; \sigma_t, \sigma_e \in \Re^+\}$, the parameter space is assumed unknown. Based on a random sample of n treatments with r_1 equal samples per treatment, we let $M_{..}$ be the over all sample mean and jointly $M_{..}$, MST and MSE constitute a complete sufficient statistic for μ, σ_t^2 and σ_e^2. Also $M_{..}$, MST and MSE are independently distributed with

$M_{..} \sim$ N $(\mu, (r_1\sigma_t^2 + \sigma_e^2)/ r_1 n)$; (n-1) $MST / (r_1\sigma_t^2 + \sigma_e^2) \sim \chi^2_{(n-1)}$ and

n (r_1-1) $MSE / \sigma_e^2 \sim \chi^2_{(n(r_1-1))}$.

Let the loss incurred in estimating μ by $M_{..}$ be

$$L(\mu, M_{..}) = A\,|\, M_{..} - \mu \,|^{\alpha} + Cn^{\beta} . \tag{2}$$

Then, the risk corresponding to the loss (2) is

$$R_n(C) = (2/\alpha)\ K^* \left[(r_1\sigma_t^2 + \sigma_e^2)/n\right]^{\alpha/2} + Cn^{\beta} , \tag{3}$$

where $K^* = (A\alpha/2)(2/r_1)^{\alpha/2}\,\Gamma((r_1+1)/2)/\Gamma(1/2)$.

And, the sample size n_0 which minimizes the risk R_n(C) is

$$n_0 = (K^*/C\beta)^{2/(\alpha+2\beta)}\left[(r_1\sigma_t^2 + \sigma_e^2)\right]^{\alpha/(\alpha+2\beta)}, \tag{4}$$

and

$$R_{n_0}(C) = (2\beta/\alpha + 1)Cn_0^{\beta} \tag{5}$$

But in the absence of any knowledge about σ_t^2 and σ_e^2, there does not exist any fixed sample size procedure which minimizes $R_n(C)$ simultaneously for all σ_t^2 and σ_e^2. Hence the problem of estimation of μ arises via the estimation of n.

In the following sections, we have proposed two multi-stage procedures to estimate μ when n, the sample size, which minimizes the risk, is unobtainable.

2 The two-stage procedure

We start with a sample of size m ($\geq$2), where m is chosen in such a manner that m=o ($C^{2/(\alpha+2\beta)}$), as C$\to$0 and $\lim_{C\to 0}$ (m/n_0) <1. Based on these m observations, we compute MST.

Then the second-stage sample size being given by

$$N = \max.\left\{m, \left[(K^*/C\beta)^{2/(\alpha+2\beta)}(MST)^{\alpha/(\alpha+2\beta)}\right]^{+} + 1\right\} , \tag{6}$$

where $[Y]^{+}$ denote the largest positive integer less than Y. Then estimate μ by $M_{..}$ using N observations. The risk associated with the 'two-stage' procedure is

$$R_N\,(\mathrm{C}) = \mathrm{C}\, n_0^{\beta} \left\{ \cdot \frac{(2/\alpha)K^{*}(r_1\sigma_t^2+\sigma_e^2)^{\alpha/2}}{Cn_0^{(\alpha+2\beta)/\alpha}} E(n_0/N)^{\alpha/2} + E(N/n_0)^{\beta} \right\} \tag{7}$$

Next, we establish second-order asymptotics for the proposed two-stage procedure.

Lemma 1. For the proposed (6) procedure

$$\underset{C\to 0}{Lim}\ (N/n_0) = 1 \quad \text{a.s.,} \tag{8}$$

and for k>0, and $C \to 0$,

$$\mathrm{E}\left(MST^{k}\right) = \left(r_1\sigma_t^2+\sigma_e^2\right)^{k} + \mathrm{o}(C^{2/(\alpha+2\beta)}) \tag{9}$$

Proof. From (6), we have the inequality

$$\left(K^{*}/C\beta\right)^{2/(\alpha+2\beta)}(MST)^{\alpha/(\alpha+2\beta)} \le \mathrm{N} \le \left(K^{*}/C\beta\right)^{2/(\alpha+2\beta)}(MST)^{\alpha/(\alpha+2\beta)} + \mathrm{m}$$

or

$$\left[MST/\left(r_1\sigma_t^2+\sigma_e^2\right)\right]^{\alpha/(\alpha+2\beta)} \le \mathrm{N}/n_0 \le \left[MST/\left(r_1\sigma_t^2+\sigma_e^2\right)\right]^{\alpha/(\alpha+2\beta)} + \mathrm{m}/n_0$$

which leads to the desired result (8) after using Kolmogrov's SLLN and a choice of m.

Since $(n-1)MST \big/ \left(r_1\sigma_t^2+\sigma_e^2\right) \sim \chi^2_{(n-1)}$, we get

$$E\left[MST^{K}\right] = \frac{\left(r_1\sigma_t^2+\sigma_e^2\right)^{K}}{\left(\frac{n-1}{2}\right)^{K}} \frac{\Gamma\left(\frac{n-1}{2}+K\right)}{\Gamma\left(\frac{n-1}{2}\right)} \tag{10}$$

and the result from O'Neill et al.(1973) that

$a^{-b}\Gamma(a+b)/\Gamma a$ = 1+o (a^{-1}), as a$\to\infty$, we obtain (9).

The main results are now obtained in the following theorem:

Theorem 1. For the two-stage procedure given by (6), as C$\to$0,

$$\mathrm{E(N)} = n_0 + \frac{1}{2} + \mathrm{o}(1), \tag{11}$$

$$\mathrm{E}(N^2) = n_0^2 + n_0 + \frac{1}{3} + \mathrm{o}(C^{\alpha/(\alpha+2\beta)}), \text{ an d} \tag{12}$$

$$\mathrm{R_g(C)} = \mathrm{C}(\alpha+2\beta)\, n_0^{\beta-2} + \mathrm{o}\left(C^{(\alpha+2)/(\alpha+2\beta)}\right) \tag{13}$$

Proof. Denoting by

$$T_m = 1 - \{ (K^{*}/C\beta)^{2/(\alpha+2\beta)}\ (MST)^{\alpha/(\alpha+2\beta)} - [\,(K^{*}/C\beta)^{2/(\alpha+2\beta)}\ (MST)^{\alpha/(\alpha+2\beta)}\,]^{+} \},$$

we can write

$$\mathrm{E(N)} = (K^{*}/C\beta)^{2/(\alpha+2\beta)} . \mathrm{E}\left\{(MST)^{\alpha/(\alpha+2\beta)}\right\} + \mathrm{E}\,(T_m)\,. \tag{14}$$

It follows from Hall (1981) that $T_m \xrightarrow{L} U(0,1)$ as $\mathrm{C}\to 0$.

Utilizing this result and (9), we obtain that, as $C \to 0$, E(N)= $n_0 + \frac{1}{2}$ +o(1).

Furthermore,

$$E(N^2) = (K^*/C\beta)^{4/(\alpha+2\beta)}\, \mathrm{E}\left\{(MST)^{2\alpha/(\alpha+2\beta)}\right\} + 2\,(K^*/C\beta)^{2/(\alpha+2\beta)}\, \mathrm{E}\left((MST)^{\alpha/(\alpha+2\beta)} T_m\right). \qquad (15)$$

And it follows from Cauchy-Schwartz Inequality that

$$Cov^2\{(\mathrm{MST})^{\alpha/(\alpha+2\beta)}, \mathrm{T_m}\} \le \mathrm{Var}\{(MST)^{\alpha/(\alpha+2\beta)}\}\, \mathrm{Var}(T_m)$$

$$= \frac{1}{12}\{E(MST)^{2\alpha/(\alpha+2\beta)} - (E(MST)^{\alpha/(\alpha+2\beta)})^2\},$$

which on applying (9) gives that, as $\mathrm{C} \to 0$, $Cov^2\{(MST)^{\alpha/(\alpha+2\beta)}, T_m\} \le$ o(1), implying that $(MST)^{\alpha/(\alpha+2\beta)}$ and T_m are asymptotically uncorrelated. Applying this result, we obtain from (15) that, as $\mathrm{C} \to 0$,

$$\mathrm{E}(N^2) = (K^*/C\beta)^{4/(\alpha+2\beta)}\,(r_1\sigma_t{}^2 + \sigma_e^2)^{2\alpha/(\alpha+2\beta)}\left\{1 + o(C^{2/(\alpha+2\beta)}\right\} + \frac{1}{3} + (K^*/C\beta)^{2/(\alpha+2\beta)}\,.$$

$$.(r_1\sigma_t{}^2 + \sigma_e^2)^{\alpha/(\alpha+2\beta)}\left\{1 + o(C^{2/(\alpha+2\beta)}\right\},$$ and (12) follows.

We can write (7) as

$R_N(C) = \mathrm{C}\, n_o{}^{\beta}\, \mathrm{E}\,[\mathrm{f}\,(\mathrm{N}/n_0)]$,

where $\mathrm{f}(x) = \{(2/\alpha)\, K^*\,(r_1\sigma_t{}^2 + \sigma_e^2)^{\alpha/2}/\mathrm{C}\, n_0^{(\alpha+2\beta)/\alpha}\}.\, x^{-\alpha/2} + x^{\beta}$.

Expanding f(x) around 'x=1' by second-order Taylor's series, we obtain for $|U-1| \le |x-1|$,

$$\mathrm{f}(x) = \mathrm{f}(1) + (x-1)\, f'(1) + \frac{(x-1)^2}{2!}\, f''(U).$$

Also, we have

$$R_N(C) = R_{n_0}(C) + \frac{Cn_0^{\beta}}{2n_0^2}\, \mathrm{E}(\mathrm{N}-n_0)\{(2/\alpha)\, K^*\,(r_1\sigma_t{}^2 + \sigma_e^2)^{\alpha/2}/\mathrm{C}\, n_0^{(\alpha+2\beta)/2}\}.$$

$$.(\alpha/2)(\alpha/2+1)U^{-(\alpha/2+2)} + \beta(\beta-1)U^{\beta-2}.$$

And for sufficiently small C, both U and U^{-1} are bounded. From (8), $\mathrm{U} \xrightarrow{a.s} 1$ as $\mathrm{C} \to 0$.

Thus, utilizing these results, (10) and (11), one gets (12) and the theorem follows.

3 The Three-Stage Procedure

Let $\eta \in (0,1)$ be specified. We start with a sample of size m(≥ 2), where m is chosen in such a manner that m=o$\left(C^{-2/(\alpha+2\beta)}\right)$ as $\mathrm{C} \to 0$ and $\underset{C\to 0}{Lim}\sup\left(m/n_0\right) < 1$. Then, denoting by $[Y]^+$- the largest positive integer <Y, we collect M-m more observations at the second stage, where,

$$\mathrm{M} = \max\left\{m, \left[\eta\left\{\left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)}.(MST)^{\alpha/(\alpha+2\beta)}\right\}\right]^+ + 1\right\}. \qquad (16)$$

Finally, at the third stage, we take N-M observations, where

$$\mathrm{N} = \max\left\{M, \left[\left\{\left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)} .(MST)^{\alpha/(\alpha+2\beta)}\right\}\right]^+ +1\right\}. \tag{17}$$

After stopping, we estimate μ by $M_{..}$.

The risk associated with the three-stage procedure is same as that given by (7).

We first establish some basic lemmas.

Lemma 2. For the three-stage procedure as $\mathrm{C} \to 0$,

$$\mathrm{E}(N) = n_0 - \frac{1}{2\eta(\alpha+2\beta)}\{3\alpha+2\beta\} + \frac{1}{2} + 0(1), \tag{18}$$

and

$$E(N^2) = n_0^2 + \frac{\{(2\eta-1)(\alpha+2\beta)-4\alpha\}}{2\eta(\alpha+2\beta)} n_0 + \left[\frac{2\alpha(\alpha-2\beta)+(3\alpha+2\beta)-\eta(\alpha+2\beta)(3\alpha-2\beta)}{2\eta^2(\alpha+2\beta)^2}\right]$$
$$+\frac{1}{3} + o\left(C^{2/(\alpha+2\beta)}\right). \tag{19}$$

Proof. By the definition of N, we have

$$\mathrm{E}(N) = \mathrm{I} + \mathrm{II}, \text{ say} \tag{20}$$

where

$$\mathrm{I} = \mathrm{E}\left[NI\left(\{M \le m\} \cup \left\{N \le \left[\eta\left\{\frac{K^*}{C\beta}\right\}^{2/(\alpha+2\beta)} (MST)^{\alpha/(\alpha+2\beta)}\right]^+ + 1\right\}\right)\right],$$

and

$$\mathrm{II} = \mathrm{E}\left[NI\left(\left[\left\{\left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)} (MST)^{\alpha/(\alpha+2\beta)}\right\}\right]^+ + 1 > M\right)\right],$$

It follows from Hall (1981) that, as $\mathrm{C} \to 0$,

$$\mathrm{I} = o(1) \tag{21}$$

Now, denoting by

$$T_M = 1 - \left\{\eta\left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)} (MST)^{\alpha/(\alpha+2\beta)} - \left[\eta\left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)} (MST)^{\alpha/(\alpha+2\beta)}\right]^+\right\},$$

we can write

$$\mathrm{II} = \left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)} E\left\{(MST)^{\alpha/(\alpha+2\beta)}\right\} + E(T_M).$$

It follows from Hall (1981), as $\mathrm{C} \to 0$, T_M tends to $U(0,1)$. Thus, as $\mathrm{C} \to 0$,

$$\mathrm{II} = \left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)} E\left\{(MST)^{\alpha/(\alpha+2\beta)}\right\} + \frac{1}{2}. \tag{22}$$

And utilizing the results that (n-1) $MST / (r_1\sigma_t^2 + \sigma_e^2) \sim \chi^2_{(n-1)}$, and a well known result of O'Neill et al.(1973) that $x^{-y}\Gamma(x+y)/\Gamma(x) = 1 + o(x^{-1})$, as $x \to \infty$, we get,

$$E(MST)^{\alpha/(\alpha+2\beta)} = (r_1\sigma_t^2 + \sigma_e^2)^{\alpha/(\alpha+2\beta)} - \frac{(r_1\sigma_t^2 + \sigma_e^2)^{\alpha/(\alpha+2\beta)}}{2(\alpha+2\beta)(\eta n_0)}(3\alpha+2\beta) + o(C^{1/(\alpha+2\beta)}) \tag{23}$$

Using (23), (22), (21) and (20), we get (18).

Furthermore, we have

$$E(N^2) = E\left\{\left[\left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)}(MST)^{\alpha/(\alpha+2\beta)}\right]^+ + 1\right\}^2$$

so that, as $C \to 0$, result (19) follows, after some algebraic adjustments.

Lemma 3. For $\eta \in (0,1)$, as $C \to 0$, $P(N \le \eta n_o) = O(m^{-r^*})$, where r^* is any positive integer.

Proof. Let $n_{1c} = [\eta n_0]^+$. It follows from the definition of N that

$$P(N \le \eta n_o) \le P\left[\left(\frac{K^*}{C\beta}\right)^{2/(\alpha+2\beta)}(MST)^{\alpha/(\alpha+2\beta)} \le \eta n_o\right]$$

$$\le P\left[\max_{m \le M \le n_{1c}}\left|MST - (r_1\sigma_t^2 + \sigma_e^2)\right| \ge (r_1\sigma_t^2 + \sigma_e^2)\left\{1 - \eta^{(\alpha+2\beta)/\alpha}\right\}\right]$$

$$= O(m^{-r^*}),$$

by Hajek-Renyi inequality(See Sen (1981)).

Lemma 4. For any $\delta(>0)$, as $C \to 0$,

$$E(N^\delta) = n_o^\delta - \frac{2\delta}{4\eta(\alpha+2\beta)}\left[2(3\alpha+1) + 2\eta(\alpha+2\beta) - (\alpha-1)(\alpha-2\beta+2)\right]n_o^{\delta-1} + o(C^{-(\delta-1)/(\alpha+2\beta)}) \quad ; \tag{24}$$

$$E(N^{-\delta}) = n_o^{-\delta} + \frac{2\delta n_o^{-(\delta+1)}}{4\eta(\alpha+2\beta)}\left[2(3\alpha+1) + 2\eta(\alpha+2\beta) - (\delta-1)(\alpha-2\beta+2)\right] + o(C^{-(\delta-1)/(\alpha+2\beta)}) \tag{25}$$

Proof. We can write

$E(N^\delta) = n_o^\delta E\left[f(\tfrac{N}{n_o})\right]$,

where f(x)= x^δ. Expanding f(x) around 'x=1' by Taylor's expansion, we obtain for $|W-1| \le |(N/n_o) - 1|$,

$$E(N^\delta) = n_o^\delta\left[1 + (\delta/n_o)E(N-n_o) + \frac{\delta(\delta-1)}{2}E\left\{\frac{(N-n_o)^2}{n_o^2}W^{\delta-2}\right\}\right].$$

Now, utilizing the fact that for δ (>0), $W^{\delta-2}$ is uniformly integrable, Lemma 2 and also using that $W \xrightarrow{a.s.} 1$, as $C \to 0$, we obtain

$$E(N^{\delta}) = n_0^{\delta}\left[1+\frac{\delta}{n_0}\left\{-\frac{1}{2\eta(\alpha+2\beta)}(3\alpha+1)+\frac{1}{2}\right\}+\frac{\delta(\delta-1)}{2n_0^2}\right.$$

$$\left.\cdot\left\{n_0^2+\frac{((2\eta-1)(\alpha+2\beta)-4\alpha)}{2\eta(\alpha+2\beta)}n_0-2n_0^2+\frac{3\alpha+1}{\eta(\alpha+2\beta)}n_0-n_0+n_0^2\right\}\right]+o(n_0^{\delta-1})$$

and (24) follows after some algebra.

Using the Taylor's expansion for $N^{-\delta}$, the fact that for δ (>0), $W^{\delta-2}$ is uniformly integrable, Lemma 2 and also utilizing that $W \xrightarrow{a.s.} 1$ as $C \to 0$, we get

$$E\left(N^{-\delta}\right) = n_0^{-\delta} - \delta n_0^{-(\delta+1)}\left[-\frac{1}{2\eta(\alpha+2\beta)}+\frac{1}{2}\right]+\frac{\delta(\delta+1)}{2}n_0^{-(\delta+2)}\left[\begin{array}{l} n_0^2+\left\{\frac{(2\eta-1)(\alpha+2\beta)-4\alpha}{2\eta(\alpha+2\beta)}\right\}n_0 \\ -2n_0^2+\frac{(3\alpha+1)}{\eta(\alpha+2\beta)}n_0-n_0+n_0^2 \end{array}\right]$$

$$+o\left(n_o^{-(\delta+1)}\right),$$

and (25) holds after some simplifications.

Theorem 3. For the three-stage procedure as $C \to 0$,

$$R_g(C) = \left(\frac{C}{\alpha}\right)n_o^{2\beta}+\left(\frac{C}{4\eta}\right)\left\{\frac{2}{(\alpha+2\beta)}+(\alpha-4\eta)-\frac{4\eta}{\alpha}n_o\right\}+o(C).$$

Proof. Using Lemma 4 and definition of $R_g(C)$, we get , as $C \to 0$,

$$R_g(C) = \left(\frac{C}{\alpha}\right)n_o^{(\alpha+2\beta)}\left[n_o^{-\alpha}+\frac{\alpha n_o^{-(\alpha+2\beta)}}{4\eta(\alpha+2\beta)}\{6\alpha+2-2\eta(\alpha+2\beta)+\alpha(\alpha+2\beta)\}+o(1)\right]$$

$$+C\left[n_o - \frac{1}{4\eta(\alpha+2\beta)}\{6\alpha+2\eta(\alpha+2\beta)\}+o(1)\right] - C(1/\alpha+1)n_o$$

$$=\left(\frac{C}{\alpha}\right)\left[n_o^{2\beta}+\frac{\alpha}{4\eta(\alpha+2\beta)}\{6\alpha+2 \quad 2\eta(\alpha+2\beta)+\alpha(\alpha+2\beta)\}\right]$$

$$+C\left[n_o - \frac{1}{4\eta(\alpha+2\beta)}\{6\alpha+2\eta(\alpha+2\beta)\}\right] - \frac{C}{\alpha}n_o - Cn_o + o(C),$$

and the theorem follows.

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