Multiple Best Choice Problems

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Abstract. We consider a generalization of the best choice problem when it is possible to make k choices. We are interested in finding a stopping rule which maximizes some expected gain. Optimal stopping rules and the value of a game are obtained.
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1 Introduction

The best choice problem or secretary problem is an important class of the theory of optimal stopping rules. The problem has been studied by many authors: Chow et al. (1971), Shiryaev (1978), Ferguson (1989). In this paper, we consider a generalization of the best choice problem — a multiple best choice problem.

We have a known number N of objects numbered $1, 2, \dots, N$, so that, say, an object numbered 1 is classified as "the best", ..., and an object numbered N is classified as "the worst". It is assumed that the objects arrive one by one in random order, i.e all $N!$ permutations are equiprobable. It is clear from comparing any two of these objects which one is better, although their actual number still remain unknown. After having known each sequential object, we either accept this object (and then a choice of one object is made), or reject it and continue observation (it is impossible to return to the rejected object). We assume that it is possible to make k choices. The aim is to find stopping rules which maximize some gain.

We can use this model to analyze some behavioral ecology problems such as sequential mate choice or optimal choice of the place of foraging. Indeed, in some species, active individuals (generally, females) sequentially mate with different passive individuals (usually males) within a single mating period (see, e.g., Gabor and Halliday (1997), Pitcher et al. (2003)). Note also that an individual can sequentially choose more than one place to forage. An active individual can either accept the item (in this case one sample has been selected), or reject it and continue the observation (it is impossible to return to the rejected item). Note also that an individual can sequentially choose more than one place to forage. The aim is to find a procedure which maximize the gain.

2 Some results of the theory of optimal multiple stopping rules

Let $y_1, y_2, \dots$ be a sequence of random variables with known joint distribution. We are allowed to observe the y_n sequentially, stopping anywhere we please. If we stop at time m_1 after observations $(y_1, \dots, y_{m_1})$, then we begin to observe another sequence y_{m_1,m_1+1}, $y_{m_1,m_1+2}, \dots$ (depending on $(y_1, \dots, y_{m_1})$), and we must solve the problem of an optimal stopping of the new sequence. If we make i stops at times $m_1, m_2, \dots, m_i$ $(1 \leqslant i \leqslant k-1)$, then we observe a sequence of random variables $y_{m_1,\dots,m_i,m_i+1}, y_{m_1,\dots,m_i,m_i+2}, \dots$ whose distribution depends on $(y_1, \dots, y_{m_1}, y_{m_1,m_1+1},$

$\dots, y_{m_1,m_2}, \dots, y_{m_1,\dots,m_i})$. Our decision to stop at times m_i $(i = 1, 2, \dots, k)$ depends solely on the values of the basic random sequence already observed and not on any future values. After k $(k \geqslant 2)$ stops we receive the gain

$$Z_{m_1,\dots,m_k} = g_{m_1,\dots,m_k}(y_1, \dots, y_{m_1,m_1+1}, \dots, y_{m_1,\dots,m_k}),$$

where $g_{m_1,\dots,m_k}$ is a known function. We are interested in finding stopping rules which maximize our expected gain.

More formally, assume that we are given:

(a) a probability space $(\Omega, \mathcal{F}, \mathbf{P})$;

(b) a nondecreasing sequence of σ-subalgebras $\{\mathcal{F}_{m_1,\dots,m_{i-1},m_i}, m_i > m_{i-1}\}$ of σ-algebra $\mathcal{F}$ such that

$$\mathcal{F}_{m_1,\dots,m_{i-1}} \subseteq \mathcal{F}_{m_1,\dots,m_i} \subseteq \mathcal{F}_{m_1,\dots,m_{i-1},m_i+1}$$

for all $i = 1, 2, \dots, k$, with $0 \equiv m_0 < m_1 < \dots < m_{i-1}$;

(c) a random process

$$\{Z_{m_1,\dots,m_{k-1},m_k}, \mathcal{F}_{m_1,\dots,m_{k-1},m_k}, m_k > m_{k-1}\}$$

for any fixed integer $m_1, \dots, m_{k-1}$, $1 \leqslant m_1 < m_2 < \dots < m_{k-1}$.

In terms of the informal background of the first paragraph in this section, we can express the σ-algebra as follows:

$$\mathcal{F}_{m_1,\dots,m_i} = \sigma(y_1, \dots, y_{m_1}, y_{m_1,m_1+1}, \dots, y_{m_1,m_2}, \dots, y_{m_1,m_2,\dots,m_i}).$$

Following Nikolaev (1977, 1999), we now give the required definitions and theorems.

Definition 1. A collection of integer-valued random variables $(\tau_1, \dots, \tau_i)$ is called an i-multiple stopping rule $(1 \leqslant i \leqslant k)$ if the following conditions hold:

a) $1 \leqslant \tau_1 < \tau_2 \dots < \tau_i < \infty$ (**P**-a.s.),

b$_j$) $\{\omega : \tau_1 = m_1, \dots, \tau_j = m_j\} \in \mathcal{F}_{m_1,\dots,m_j}$ for all $m_j > m_{j-1} > \dots > m_1 \geqslant 1; j = 1, 2, \dots, i$.

Definition 2. A k-multiple stopping rule with $k > 1$ is called a multiple stopping rule.

We use the following notation, where ξ represents as arbitrary random variable:

$$(m)_i = (m_1, m_2, \dots, m_i),\ (m)_1 = m_1,\ \mathbf{E}_{(m)_i}\xi = \mathbf{E}(\xi \mid \mathcal{F}_{(m)_i}),$$
$$A_{(m)_i}\xi = \mathbf{E}_{(m)_i}\left(\sup_{m_{i+1}} \mathbf{E}_{(m)_{i+1}}\left(\dots\left(\sup_{m_{k-1}} \mathbf{E}_{(m)_{k-1}}\xi\right)\dots\right)\right).$$

The following condition is needed for the existence of all considered expectations.

$$(A^+):\ \mathbf{E}\left(\sup_{m_1} A_{(m)_1}\left(\sup_{(m)_k} Z_{(m)_k}\right)\right) < +\infty.$$

We assume that condition (A^+) is satisfied for the $Z_{(m)_k}$.

Let S_m be a class of multiple stopping rules $\tau = (\tau_1, \dots \tau_k)$ such that $\tau_1 \geqslant m$ (**P**-a.s.).

Definition 3. The function

$$v_m = \sup_{\tau \in S_m} \mathbf{E}Z_\tau$$

is called the *m-value of the game*. In particular, if $m = 1$ then $v = v_1$ is called the *value of the game*.

Definition 4. A multiple stopping rule $\tau^* \in S_m$ is called an *optimal multiple stopping rule* in S_m if $\mathbf{E}Z_{\tau^*}$ exists and $\mathbf{E}Z_{\tau^*} = v_m$.

The aforementioned condition (A^+) ensures the finiteness of v_m and the existence of $\mathbf{E}Z_\tau$ for all $\tau \in S_m$. The problem consists of finding an optimal multiple stopping rule and an m-value of the game v_m.

The sequences $\{V_{(m)_i}\}$ and $\{X_{(m)_i}\}$, $i = 1, 2, \ldots, k$ are needed for constructing the multiple stopping rules τ^*. Let $T_{(m)_i}$ be a class of i-multiple stopping rules $(\tau)_i = (\tau_1, \ldots, \tau_i)$ $(i = 1, 2, \ldots, k)$ with $\tau_1 = m_1, \ldots, \tau_{i-1} = m_{i-1}$, $\tau_i \geqslant m_i$ (**P**-a.s.). Let $T_{(m)_1} \equiv T_{m_1}$ denote the class of all stopping times τ_1 such that $\tau_1 \geqslant m_1$ (**P**-a.s.). We set $X_{(m)_k} = Z_{(m)_k}$ and define by backward induction on i from $i = k$:

$$V_{(m)_i} = \operatorname{ess} \sup_{(\tau)_i \in T_{(m)_i}} \mathbf{E}_{(m)_i} X_{(\tau)_i},$$
$$X_{(m)_{i-1}} = \mathbf{E}_{(m)_{i-1}} V_{(m)_{i-1}, m_{i-1}+1}, \quad i = k, k-1, \ldots, 1,$$

where $X_0 \equiv 0$.

We emphasize that most of the statements in this section are valid almost surely. We shall make no mention of this in what follows.

Let us now establish some properties of the sequences $\{V_{(m)_i}\}$ and $\{X_{(m)_i}\}$. It follows from results of the general theory of optimal stopping (see, e.g., Chow et al. (1971), Haggstrom (1966)) that $V_{(m)_i}$ satisfies the recursion equation

$$V_{(m)_i} = \max\{X_{(m)_i}, \mathbf{E}_{(m)_i} V_{(m)_{i-1}, m_i+1}\}.$$

The following theorem gives the existence conditions and the structure of an optimal multiple stopping rule in S_m.

Theorem 1. *Let condition (A^+) be satisfied. We put*

$$\tau_i^* = \inf\{m_i > m_{i-1} : V_{(m)_i} = X_{(m)_i}\}$$

for $i = 1, 2, \ldots, k$ on the set $D_{i-1} = \{\omega : \tau_1^* = m_1, \ldots, \tau_{i-1}^* = m_{i-1}\}$, where it is assumed that $\tau_i^*(\omega) = +\infty$ on $\{\omega : \tau_{i-1}^*(\omega) = +\infty\}$, $m_0 = m - 1$, and $D_0 = \Omega$. In that case, if the random vector $\tau^* = (\tau_1^*, \ldots, \tau_k^*)$ is finite with probability one, then $\tau^* \in S_m$ is an optimal multiple stopping rule.

The following theorem gives the characterization of the m-value v_m by means of the sequence $\{V_m\}$.

Theorem 2. *If condition (A^+) holds, then $v_m = \mathbf{E}V_m$.*

We now consider a finite case. Let

$$\{Z_{(m)_k}, 1 \leqslant m_1 \leqslant N_1, m_1 < m_2 < N_2(m_1), \ldots,$$
$$m_{k-1} < m_k \leqslant N_k(m_1, \ldots, m_{k-1})\}$$

be a family of random variables, where N_1, $N_i(\cdot)$ $(i = 2, \ldots, k)$ are natural numbers. As in the general theory of optimal stopping (see Chow et al. (1971)), we define the sequence $V_{(m)_i}$ by backward induction from the recursion equations

$$V_{(m)_{i-1}, N_i(m_1, \ldots, m_{i-1})} = X_{(m)_{i-1}, N_i(m_1, \ldots, m_{i-1})}, \tag{1}$$
$$V_{(m)_i} = \max\{X_{(m)_i}, \mathbf{E}_{(m)_i} V_{(m)_{i-1}, m_i+1}\} \tag{2}$$

for $1 \leqslant m_1 \leqslant N_1, \ldots, m_{i-1} < m_i \leqslant N_i(m_1, \ldots, m_{i-1})$. As before, $X_{(m)_k} = Z_{(m)_k}$.

Using Theorem 1, we define the optimal multiple stopping rule τ^*. From Theorem 2, (1), and (2), we obtain the value v_m.

3 The Multiple Best Choice Problem

Suppose the gain is the probability of choosing k best objects. Denote by $(a_1, a_2, \ldots, a_N)$ any permutation of numbers $(1, 2, \ldots, N)$, 1 corresponds to the best object, N corresponds to the worst one. If a_i is the m-th object in order on quality among $(a_1, a_2, \ldots, a_i)$, we write $y_i = m$ for all $i = 1, 2, \ldots, N$, a_i is called the *absolute rank*, and y_i is called the *relative rank*.

Let $(i_1, \ldots, i_k)$ be any permutation of numbers $1, 2, \ldots, k$. A rule $\tau^* = (\tau_1^*, \ldots, \tau_k^*)$, $1 \le \tau_1^* < \tau_2^* < \cdots < \tau_k^* \le N$ is an optimal rule if

$$\mathbf{P}\Big\{ \bigcup_{(i_1,\ldots,i_k)} \{a_{\tau_1^*} = i_1, \ldots, a_{\tau_k^*} = i_k\} \Big\} = \sup_\tau \mathbf{P}\Big\{ \bigcup_{(i_1,\ldots,i_k)} \{a_{\tau_1} = i_1, \ldots, a_{\tau_k} = i_k\} \Big\} = \mathbf{P}_N^*, \tag{3}$$

where $\tau = (\tau_1, \ldots, \tau_k)$. We are interested in finding the optimal rule $\tau^* = (\tau_1^*, \ldots, \tau_k^*)$.

By $Z_{(m)_k}^{(i)_k} = Z_{m_1,\ldots,m_k}^{i_1,\ldots,i_k}$ denote a conditional probability of event $\{a_{m_1} = i_1, \ldots, a_{m_k} = i_k\}$ with respect to σ-algebra $\mathcal{F}_{(m)_k}$, generated by observations $(y_1, \ldots, y_{m_k})$, and put

$$Z_{(m)_k} = \sum_{(i_1,\ldots,i_k)} Z_{(m)_k}^{(i)_k}.$$

Using (3), we get the value of the game v

$$\mathbf{P}_N^* = \mathbf{E} Z_{\tau^*} = \sup_\tau \mathbf{E} Z_\tau = v.$$

Thus we reduce the best choice problem of k objects to the problem of multiple stopping of the random sequence $Z_{(m)_k}$ (for further details see Nikolaev M.L. (1998)).

As was shown in Nikolaev (1977, 1998), the solution of this problem is the following optimal strategy: there exists a set $\boldsymbol{\pi}^* = (\pi_1^*, \ldots, \pi_k^*)$, $1 \le \pi_1^* < \cdots < \pi_k^* \le N$ such that

- it is necessary to skip first $\pi_1^* - 1$ objects, and then we stop on the first object, which is better than all precursors, or on the $(N-k+1)$-th object, if the best one does not appear by the moment $N-k+1$;
- at second time we stop on the first object, which is better than all precursors, or worse than one object (if we already have observed $\pi_2^* - 1$ objects), if any, or, otherwise, on $(N-k+2)$-th object;
- the third choice should be made on the first object, which is better than all precursors, or worse than one object (if we already have observed $\pi_2^* - 1$ objects) or worse than two objects (if we already have observed $\pi_3^* - 1$ objects), if any, or on $(N-k+3)$-th object etc.

More formally,

$$\begin{aligned}
\tau_1^* &= \min\{m_1 \ge \pi_1^* : y_{m_1} = 1\}, \\
\tau_i^* &= \min\big[\min\{m_i > m_{i-1} : y_{m_i} = 1\}, \\
&\qquad \min\{m_i > m_{i-1} : m_i \ge \pi_2^*, y_{m_i} = 2\}, \\
&\qquad \ldots, \min\{m_i > m_{i-1} : m_i \ge \pi_i^*, y_{m_i} = i\}\big]
\end{aligned}$$

on the set $F_{i-1} = \{\omega : \tau_1^* = m_1, \ldots, \tau_{i-1}^* = m_{i-1}\}$, $i = 2, \ldots, k$, $F_0 = \Omega$.

Example 1. If $N = 5$, $k = 2$, then $v = 0.3333$, $\pi_1^* = 2$, $\pi_2^* = 4$. Consider the following random permutation $a_1, \ldots, a_5$: 3, 2, 4, 5, 1. Then we observe the sequence of $y_1, \ldots, y_5$: 1,1,3,4,1. According to the optimal rule we stop on the second object $(m_1 = 2, y_2 = 1)$ and on the fifth one $(m_2 = 5, y_5 = 1)$. Thus we choose the best and the second best objects.

4 Generalizations of the multiple best choice problem

We can generalize the best multiple choice problem to the case when the gain is the probability of choosing k objects with given ranks $r_1, r_1, \ldots, r_k$, $1 \le r_1 < \ldots < r_k \le N$ (Nikolaev et al. (2007)).

Let $(l)_k = (l_1, \ldots, l_k)$ denote a permutation of the integer numbers $r_1, r_2, \ldots, r_k$. A rule $\tau^* = (\tau_1^*, \ldots, \tau_k^*)$, $1 \le \tau_1^* < \tau_2^* < \cdots < \tau_k^* \le N$ is an optimal rule if

$$\mathbf{P}\left\{\bigcup_{(l)_k}\{a_{\tau_1^*} = l_1, \ldots, a_{\tau_k^*} = l_k\}\right\} = \sup_{\tau} \mathbf{P}\left\{\bigcup_{(l)_k}\{a_{\tau_1} = l_1, \ldots, a_{\tau_k} = l_k\}\right\} = \mathbf{P}_N^*, \qquad (4)$$

$\tau = (\tau_1, \ldots, \tau_k)$. We are interested in finding the optimal rule $\tau^* = (\tau_1^*, \ldots, \tau_k^*)$. In the same way, we define the sequence $Z_{(m)_k}$ and the value of the game v.

Put

$$\begin{aligned}
B_0 = &\bigcap_{s=1}^{n_0}\{\omega : y_{t_{0,s}} = j_{0,s}, y_{t_{0,s}+1} > s, \ldots, y_{t_{0,s+1}-1} > s\}, \\
&1 \le t_{0,1} < \cdots < t_{0,n_0} < t_{0,n_0+1} = m_1, \\
B_i = &\{\omega : y_{m_i} = j_i, y_{m_i+1} > n_0 + \cdots + n_{i-1} + i - 1, \ldots, \\
&y_{m_i+t_{i,1}-1} > n_0 + \cdots + n_{i-1} + i - 1\} \\
&\bigcap_{s=1}^{n_i}\{\omega : y_{m_i+t_{i,s}} = j_{i,s}, y_{m_i+t_{i,s}+1} > n_0 + \cdots + n_{i-1} + i + s - 1, \ldots, \\
&y_{m_i+t_{i,s+1}-1} > n_0 + \cdots + n_{i-1} + i + s - 1\}, \\
&1 \le t_{i,1} < \cdots < t_{i,n_i} < t_{i,n_i+1} = m_{i+1} - m_i, \quad i = 1, \ldots, k-1,
\end{aligned}$$

where $(j_1, \ldots, j_k)$, $1 \le j_i \le r_k - k + i$, is a set of the relative ranks from $R = \{r_1, \ldots, r_k\}$, $(j_{i,1}, \ldots, j_{i,n_i})$, $i = 0, 1, \ldots, k-1$ is a set of the relative ranks from $R' = \{1, 2, \ldots, r_k\} \setminus R$.

Then

$$Z_{(m)_k}^{(l)_k} = \sum \mathbf{P}\left\{a_{m_1} = l_1, \ldots, a_{m_k} = l_k \mid \{\omega : y_{m_k} = j_k\} \bigcap_{i=0}^{k-1} B_i\right\} I(y_{m_k} = j_k) I\left(\bigcap_{i=0}^{k-1} B_i\right),$$

where we sum up all ordered arrangements of $(r_k - k - n_k)$, $n_k = 0, 1, \ldots, r_k - k$, elements from the set R' in k intervals: in front of object l_1, between object l_1 and object l_2, …, between object l_{k-1} and object l_k. The rest of n_k elements from the set R' are situated after object l_k, so they do not influence on the event $\{\omega : y_{m_k} = j_k\} \bigcap_{i=0}^{k-1} B_i$. Since $y_1 = 1$, $y_2 = 1$ or 2, …, $y_i = 1, 2, \ldots, i$, then $I(B_0) = 1$. Hence, using independence of the relative ranks $y_1, \ldots, y_N$, we obtain

$$Z_{(m)_k}^{(l)_k} = \sum C_{r_k-k}^{n_k} A_{N-m_k}^{n_k} \frac{m_k(m_k-1)\ldots(m_k - r_k + n_k + 1)}{N(N-1)\ldots(N - r_k + 1)} I(y_{m_k} = j_k) I\left(\bigcap_{i=1}^{k-1} B_i\right),$$

where $C_{r_k-k}^{n_k}$ is the number of n_k-combinations from a set with $r_k - k$ elements, $A_{N-m_k}^{n_k}$ is the number of n_k-permutations from a set with $N - m_k$ elements.

Using Theorem 1, we obtain the optimal multiple stopping rule, which can be described as follows

$$\begin{aligned}
&\tau_1^* = \min\{m_1 : y_{m_1} \in \Gamma_{1,m_1}\}, \quad \Gamma_1 = (\Gamma_{1,1}, \ldots, \Gamma_{1,N-k+1}), \\
&\Gamma_{1,s} \subseteq \{1, \ldots, s\} \cap \{1, \ldots, r_k - k + 1\}, \quad s = 1, \ldots, N - k, \\
&\Gamma_{1,N-k+1} = \{1, 2, \ldots, N - k + 1\}, \\
&\tau_i^* = \min\{m_i > m_{i-1} : y_{m_i} \in \Gamma_{i,m_i}(y_{m_1}, y_{m_1+1}, \ldots, y_{m_i-1})\}, \\
&\Gamma_i = (\Gamma_{i,i}, \ldots, \Gamma_{i,N-k+i}), \quad \Gamma_{i,N-k+i} = \{1, 2, \ldots, N - k + i\}, \\
&\Gamma_{i,s} \subseteq \{1, \ldots, s\} \cap \{1, \ldots, r_k - k + i\}, \quad s = i, \ldots, N - k + i - 1.
\end{aligned}$$

The "stopping" sets $\Gamma_1, \ldots, \Gamma_k$ can be defined by backward induction. Since the structure of the sets solely depends on the values $r_1, \ldots, r_k$, finding of the optimal stopping rules and the value of the game is a problem for each particular case.

We also consider a generalization of the Gusein-Zade problem (see Gusein-Zade (1966)), whose gain is the probability of choosing one object from the l best objects. The optimal stopping rules are obtained if the gain is the probability of choosing k objects from l best ones.

The solution of this problem is the following strategy: there exist sets $\boldsymbol{\pi}^{(1)} = (\pi_1^{(1)}, \pi_2^{(1)}, \ldots, \pi_{l-k+1}^{(1)})$, $\boldsymbol{\pi}^{(2)} = (\pi_1^{(2)}, \pi_2^{(2)}, \ldots, \pi_{l-k+2}^{(2)})$, $\ldots$, $\boldsymbol{\pi}^{(k)} = (\pi_1^{(k)}, \pi_2^{(k)}, \ldots, \pi_l^{(k)})$ such that

- it is necessary to skip first $\pi_1^{(1)} - 1$ objects, and then we stop on the first object, which is better than all precursors, or the second best object, if we already have observed $\pi_2^{(1)} - 1$, and so on, or on object, which is worse than $l - k$ precursors, if we already have observed $\pi_{l-k+1}^{(1)} - 1$, or on the $(N - k + 1)$-th object;
- at second time we stop on the first object, which is better than all precursors, if we have passed $\pi_1^{(2)} - 1$ objects, or worse than one object (if we already have observed $\pi_2^{*} - 1$ objects), and so on, or on object, which is worse than $l - k + 1$ precursors, if we already have observed $\pi_{l-k+2}^{(2)} - 1$, or, otherwise, on $(N - k + 2)$-th object;
- $\ldots$
- the k-th choice should be made on the first object, which is better than all precursors, if we have observed $\pi_1^{(k)} - 1$ objects, or worse than one object (if we already have observed $\pi_2^{(k)} - 1$ objects) or worse than two objects (if we already have observed $\pi_3^{(k)} - 1$ objects), and so on, or on object, which is worse than $l - 1$ precursors, if we already have observed $\pi_l^{(k)} - 1$, if any, or on N-th object.

The problem of choosing two objects from three best ones is considered in detail by Polushina (2007). Table 1 shows the values of the game v for different N both for this problem and for the problem of choosing two best objects.

Table 1. The value of the game v.

N	3	4	5	6	7	8	9
Choice of 2 objects from 3 best ones	1.0000	0.7083	0.6333	0.5917	0.5530	0.5250	0.5059
Choice of 2 best objects	0.5000	0.3333	0.3333	0.3139	0.2956	0.2800	0.2739

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