Unspecified distributions in disorder problem

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Abstract. We register a stochastic sequence affected by one or two disorders (two cases are considered). Monitoring is made in the circumstances when not full information about distributions between changes is available. Our aim is to detect the disorder or to localize the segment between changes (depending on the case). Both problems are transformed to optimal stopping of observed sequence and formulas of optimal decision functions are derived.
Keywords. Disorder problem, sequential detection, optimal stopping, Markov process, change point.

1 Introduction

The paper is focused on sequential detection using Bayesian approach. Disorder problem in this framework was formulated by A.N. Kolmogorov at the end of 50's of previous century and solved by Shiryayev Shiryaev (1961). The next turning point is paper of Peskir and Shiryaev (2002) where authors provide complete solution of basic problem. From this time many publications provide new solutions and generalizations in the area of sequential detection. Some of them are Karatzas (2003), Bayraktar et al. (2005). For discrete time case there are known publications of Bojdecki (1979), Bojdecki and Hosza (1984), Moustakides (1998), Szajowski (1996) and Yoshida (1983). One of direction focuses on models which assume uncertainty about distribution before or/and after the change. The example is Dube and Mazumdar (2001) with application in detection of traffic anomalies in networks or Sarnowski and Szajowski (2008). Our paper also contributes to this direction of research. We present solutions of two models which assume single and double disorder with unspecified distributions of observed sequences.

Proofs of theses contained in this paper are fully presented in Sarnowski and Szajowski (2009) and Sarnowski and Szajowski (2008).

2 Unspecified distributions in single disorder problem

2.1 Problem formulation

We register process $X = \{X_n, n \in \mathbb{N}\}$. At random moment θ the change occurs in distribution of $\{X_n\}$. Our knowledge about densities before and after the change θ is limited to the information about sets of possible conditional densities: $\{f_x^{0,i}(y), i \in B_1 = \{1, \ldots, l_1\}\}$ and $\{f_x^{1,j}(y), j \in B_2 = \{1, \ldots, l_2\}\}$ respectively. Densities are given with respect to measure μ_x. We know also transition probabilities between densities: $b_{ij} = \mathbf{P}_x(\beta_1 = i, \beta_2 = j)$, $i \in B_1$, $j \in B_2$ as well as a priori distributions:

$$\mathbf{P}(\theta = j) = \pi \cdot \mathbf{1}_{\{j=1\}} + (1-\pi)p^{j-2}q \cdot \mathbf{1}_{\{j>1\}} \tag{1}$$

$$\mathbf{P}(\theta = k, \beta_1 = i, \beta_2 = j) = \pi_{ij} b_{ij} \cdot \mathbf{1}_{\{k=1\}} + (1-\pi_{ij})p_{ij}^{k-2} q_{ij} b_{ij} \cdot \mathbf{1}_{\{k>1\}} \tag{2}$$

where $\pi \in [0,1]$, $p = 1-q \in (0,1)$, $i \in B_1, j \in B_2$, $\pi_{ij} \in [0,1]$, $b_{ij} = \mathbf{P}(\beta_1 = i, \beta_2 = j) \in [0,1]$, $p_{ij} = 1 - q_{ij} \in (0,1)$. Under these conditions the following model is assumed: $X_n = X_n^{0,i} \cdot \mathbf{1}_{\{\theta > n,\, \beta_1 = i\}} + X_n^{1,j} \cdot \mathbf{1}_{\{\theta \le n,\, \beta_2 = j\}}$, where $\{X_n^{0,i}\}$ and $\{X_n^{1,j}\}$ are Markov processes with values in space $(\mathbb{E}, \mathcal{B})$, $\mathbb{E} \subset \mathbb{R}$. We wish to detect the change as close θ as possible. For $\mathcal{S}$ - the set of stopping times w.r.t. $\{\mathcal{F}_n\}_{n=0}^{\infty}$, where $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ we search for stopping time $\tau^* \in \mathcal{S}$ such that, for each $x \in \mathbb{E}$

$$\mathbf{P}(|\theta - \tau^*| \le d) = \sup_{\tau \in \mathcal{S}} \mathbf{P}(|\theta - \tau| \le d). \tag{3}$$

2.2 Existence of solution

For $X_0 = x$ a.s. and given $\phi = (\overline{\pi}, \overline{b}, x)$ let us define:

$$\begin{aligned} &Z_n = \mathbf{P}^\phi(|\theta - n| \le d \mid \mathcal{F}_n),\ Y_n = \operatorname{esssup}_{\{\tau\in\mathfrak{S}^{\mathrm{X}},\ \tau\ge \mathrm{n}\}}\mathbf{P}^\phi(|\theta-\mathrm{n}|\le \mathrm{d}\mid\mathcal{F}_{\mathrm{n}});\ \mathrm{n}=1,2,\ldots, \\ &\tau_0 = \inf\{n : Z_n = Y_n\} \end{aligned} \quad (4)$$

Lemma 1 *Stopping time τ_0 defined by (4) is a solution of (3).*

Hence, the solution exists. Moreover we need at least d observations to detect disorder in optimal way:

Lemma 2 *Let τ be a stopping time in the problem (3). Then $\tilde{\tau} = \max(\tau, d+1)$ is at least as good as τ (in the sense of (3)).*

2.3 Solution of the problem

It will be convenient to introduce following notations:

$$\begin{aligned} \underline{x}_{0,n} &= (x_k, x_{k+1}, ..., x_{n-1}, x_n);\ k \le n \\ \overline{\alpha} &= (\alpha_{11}, \ldots, \alpha_{1l_2}, \ldots, \alpha_{l_1 1}, \ldots, \alpha_{l_1 l_2}) \\ \widehat{f}^0_x(y) &= (\underbrace{f^{0,1}_x(y), \ldots, f^{0,1}_x(y)}_{l_2 \text{ times}}, \ldots, \underbrace{f^{0,l_1}_x(y), \ldots, f^{0,l_1}_x(y)}_{l_2 \text{ times}});\ x, y \in \mathbb{E} \\ L^{i,j}_m(\underline{x}_{k,n}) &= \prod_{r=k+1}^{n-m} f^{0,i}_{x_{r-1}}(x_r) \prod_{r=n-m+1}^{n} f^{1,j}_{x_{r-1}}(x_r);\quad k \le m \le n-k. \end{aligned}$$

We hold convention: $\prod_{r=m_1}^{m_2} u_r = 1$ for $m_1 > m_2$ and $u_r \in \Re$.
Let us also define operation "$\circ$". For vectors $\overline{\alpha}$ and $\overline{\beta}$ we put:

$$\overline{\alpha} \circ \overline{\beta} = (\alpha_{11}\beta_{11}, \ldots, \alpha_{1l_2}\beta_{1l_2}, \ldots, \alpha_{l_1 1}\beta_{l_1 1}, \ldots, \alpha_{l_1 l_2}\beta_{l_1 l_2}).$$

The key part of solution are posterior processes:

$$B^{i,j}_n = \mathbf{P}^\phi(\beta_1 = i, \beta_2 = j|\mathcal{F}_n);\ \mathit{\Pi}^{i,j}_n = \mathbf{P}^\phi(\theta \le n|\beta_1 = i, \beta_2 = j, \mathcal{F}_n);\ n \in \mathbb{N}; i \in B_1; j \in B_2 \quad (5)$$

In consequence of already introduced notation, vectors $\overline{\mathit{\Pi}}_n$, $\overline{B}_n$ represent:

$$\begin{aligned} \overline{\mathit{\Pi}}_n &= \left(\mathit{\Pi}^{11}_n, \ldots, \mathit{\Pi}^{1l_2}_n, \ldots, \mathit{\Pi}^{l_1 1}_n, \ldots, \mathit{\Pi}^{l_1 l_2}_n\right) \\ \overline{B}_n &= \left(B^{11}_n, \ldots, B^{1l_2}_n, \ldots, B^{l_1 1}_n, \ldots, B^{l_1 l_2}_n\right) \end{aligned}$$

Using (5) we are able to cast initial problem (3) to the case of stopping Random Markov Function with special payoff.

Lemma 3 *Let $\xi_n = (\underline{X}_{n-d-1,n}, \overline{\mathit{\Pi}}_n, \overline{B}_n)$. Process $\{\xi_n\}$ constitutes Random Markov Function.*

With help of Bayes' formula we express initial payoff as expectation of function depending on ξ_n components:

$$h(\underline{x}_{1,d+2}, \overline{\gamma}, \overline{\delta}) = \sum_{i,j}\left(1 - p^d_{ij} + q_{ij}\sum_{k=1}^{d+1}\frac{L^{i,j}_k(\underline{x}_{1,d+2})}{p^k_{ij}\, L^{i,j}_0(\underline{x}_{1,d+2})}\right)(1-\gamma_{ij})\delta_{ij}$$

replacing $(\underline{x}_{1,d+2}, \overline{\gamma}, \overline{\delta})$ by ξ_n. Thanks to lemma 3 and reformulation of payoff function we construct the solution using standard tools of optimal stopping theory for markovian processes.

To solve reduced problem, for Borel function $u : \mathbb{E}^{d+2} \times [0,1]^{l_1 l_2} \times [0,1]^{l_1 l_2} \longrightarrow \Re$ let us define operators:

$$\mathbf{T}u(\underline{x}_{1,d+2},\overline{\gamma},\overline{\delta}) = \mathbf{E}^{\phi}\left[u(\underline{X}_{n-d,n+1},\overline{\Pi}_{n+1},\overline{B}_{n+1}) \mid \underline{X}_{n-1-d,n} = \underline{x}_{1,d+2},\overline{\Pi}_n = \overline{\gamma},\overline{B}_n = \overline{\delta}\right]$$
$$\mathbf{Q}u(\underline{x}_{1,d+2},\overline{\gamma},\overline{\delta}) = \max\{u(\underline{x}_{1,d+2},\overline{\gamma},\overline{\delta}),\mathbf{T}u(\underline{x}_{1,d+2},\overline{\gamma},\overline{\delta})\},\ k \geq 1$$

With help of optimal stopping theory (c.f. Shiryayev (1978)), we infer that the solution of the problem (3) is Markov time $\tau^* = \inf\{n \geq d+1 : h(\xi_n) \geq \lim_{k\to\infty} Q_x^k h(\xi_n)\}$. The following theorem describes optimal stopping rule:

Theorem 1 *The solution of (3) is given by:*

$$\tau^* = \inf\left\{n \geq d+1 : \sum_{i,j}\left(1 - p_{ij}^d + q_{ij}\sum_{k=1}^{d+1}\frac{L_k^{i,j}(\underline{X}_{n-1-d,n})}{p_{ij}^k L_0^{i,j}(\underline{X}_{n-1-d,n})}\right)(1-\Pi_n^{i,j})B_n^{i,j} \geq r^*(\underline{X}_{n-1-d,n},\overline{\Pi}_n,\overline{B}_n)\right\} \tag{6}$$

where $r^*(\underline{X}_{n-1-d,n},\overline{\Pi}_n,\overline{B}_n) = \lim_{k\longrightarrow\infty} r_k(\underline{X}_{n-1-d,n},\overline{\Pi}_n,\overline{B}_n)$. *Functions* r_k *are obtained in recursive way:*

$$r_0(\underline{X}_{n-1-d,n},\overline{\Pi}_n,\overline{B}_n) = \sum_{i,j}\left(1 - p_{ij}^d + q_{ij}\sum_{m=1}^{d+1}\frac{L_{m-1}^{i,j}(\underline{X}_{n-d,n})}{p_{ij}^m L_0^{i,j}(\underline{X}_{n-d,n})}\right)p_{ij}(1-\Pi_n^{i,j})B_n^{i,j}.$$

$$\begin{aligned} &r_{k+1}(\underline{X}_{n-1-d,n},\overline{\Pi}_n,\overline{B}_n) \\ &= \int_{\mathbb{E}} \max\left\{\sum_{i,j}\left(1 - p_{ij}^d + q_{ij}\sum_{m=1}^{d+1}\frac{L_m^{i,j}((\underline{X}_{n-d,n},y))}{p_{ij}^m L_0^{i,j}((\underline{X}_{n-d,n},y))}\right)f_{X_n}^{0,i}(y)p_{ij}(1-\Pi_n^{i,j})B_n^{i,j},\right. \\ &\qquad \left. r_k(\underline{X}_{n-d,n},y,\overline{\Pi}_n,\overline{p}\circ\widehat{f}_{X_n}^0(y)\circ\overline{B}_n)\right\} d\mu_{X_n}(y) \end{aligned} \tag{7}$$

3 Unspecified distributions in double disorder problem

3.1 Problem formulation

In this model the observed process $X = \{X_n, n \in \mathbb{N}\}$ is obtained by switching at random instants θ_1 and θ_2 between Markov processes with values in space $(\mathbb{E},\mathcal{B})$. The second segment of observed sequence has unspecified distribution, but is chosen randomly using r.v. ε from the set of possible distributions:

$$X_n = X_n^1 \cdot \mathbf{1}_{\{\theta_1 > n\}} + X_n^{2,i} \cdot \mathbf{1}_{\{\theta_1 \leq n < \theta_2,\ \varepsilon = i\}} + X_n^3 \cdot \mathbf{1}_{\{\theta_2 \leq n\}}. \tag{8}$$

Probability structures of X_n^1, $X_n^{2,i}$, X_n^3 are determined by measures $\mu_x^1(dy) = 1\cdot\mu_x(dy)$, $\mu_x^{2,i}(dy) = f_x^i(y)\mu_x(dy),\ i = 1,...,d$, $\mu_x^3(dy) = g_x(y)\mu_x(dy)$. Variable ε is independent on θ_1, θ_2 and $\mathbf{P}(\varepsilon = i) = e_i$, $i = 1,2,...,d$; $d < \infty$, $\sum_{i=1}^d e_i = 1$. Moreover

$$P(\theta_1 = j) = p_1^{j-1}q_1,\ P(\theta_2 = k \mid \theta_1 = j) = p_2^{k-j-1}q_2\ ;\ k > j,\ j = 1,2,... \tag{9}$$

The aim is to stop the sequence between the disorders θ_1 and θ_2. We want to find stopping time $\tau^* \in \mathcal{S}$, where $\mathcal{S}$ is the set of all stopping times with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbf{N}}$, such that:

$$\mathbf{P}(\tau^* < \infty, \theta_1 \leq \tau^* < \theta_2) = \sup_{\tau\in\mathcal{S}}\mathbf{P}(\tau < \infty, \theta_1 \leq \tau < \theta_2) \tag{10}$$

3.2 Existence of solution

As in previous problem, for $X_0 = x$ let us define: $Z_n = P_x(\theta_1 \leq n < \theta_2 \mid \mathcal{F}_n)$ and $Y_n = \operatorname{esssup}_{\{\tau\in\mathcal{T},\ \tau\geq\mathrm{n}\}}\mathrm{P_x}(\theta_1 \leq \tau < \theta_2 \mid \mathcal{F}_\mathrm{n})$ for $n = 0,1,2,\ldots$ as well as:

$$\tau_0 = \inf\{n : Z_n = Y_n\}. \tag{11}$$

Lemma 4 *The stopping time* τ_0 *defined by formula (11) is the solution of problem (10).*

3.3 Solution of the problem

Optimal strategy for detection of data segment strongly bases on posterior processes

$$\Pi_n^1 = \mathbf{P}_x(\theta_1 \leq n|\mathcal{F}_n),\ \Pi_n^2 = \mathbf{P}_x(\theta_2 \leq n|\mathcal{F}_n),\ n \in \mathbb{N} \tag{12}$$

Posterior processes enable us to reduce the initial problem to the case of stopping Random Markov Function with appropriate payoff.

Lemma 5 *System $\xi_n = (X_{n-1}, X_n, \Pi_n^1, \Pi_n^2)$ forms Random Markov Function.*

The reward function is given by $h(x,y,\alpha,\beta) = \alpha - \beta$, where $x, y \in \mathbb{E}$, $\alpha, \beta \in [0,1]$, and it results from the fact that $P_x(\theta_1 \leq n < \theta_2 \mid \mathcal{F}_n) = \Pi_n^1 - \Pi_n^2$. Thanks to Lemma 5 we solve the problem applying optimal stopping theory of Markov processes.

For any Borel function $v : \mathbf{E}^2 \times [0,1]^{d+1} \longrightarrow [0,1]$ let us define two operators:

$$T_x v(y,z,\alpha,\beta) = E_x(v(X_n, X_{n+1}, \Pi_{n+1}^1, \Pi_{n+1}^2) \mid X_{n-1} = y, X_n = z, \Pi_n^1 = \alpha, \Pi_n^2 = \beta)$$
$$Q_x v(y,z,\alpha,\beta) = \max\{v(y,z,\alpha,\beta), T_x v(y,z,\alpha,\beta)\}$$

The basic equation for optimal stopping time, i.e $\tau^* = \inf\{h(X_n, X_{n+1}, \Pi_{n+1}^1, \Pi_{n+1}^2) \geq \lim_{k\to\infty} Q_x^k h(X_n, X_{n+1}, \Pi_{n+1}^1 \Pi_{n+1}^2)\}$ is clarify by the following theorem:

Theorem 2 **(a)** *The solution of problem (10) is given by:*

$$\tau^* = \inf\{n : (X_n, X_{n+1}, \Pi_{n+1}^1, \Pi_{n+1}^2) \in B^*\} \tag{13}$$

Set B^* is of the form:

$$\begin{aligned} B^* = \{(y,z,\alpha,\beta) : (\alpha - \beta) &\geq (1-\alpha) \\ &\times \left[p_1 \int_{\mathbf{E}} R^*(y,u,\Pi^1(y,u,\alpha,\beta), \Pi^2(y,u,\alpha,\beta))\mu_y(du) \right. \\ &+ \left. q_1 \int_{\mathbf{E}} S^*(y,u,\Pi^1(y,u,\alpha,\beta), \Pi^2(y,u,\alpha,\beta)) \left\langle \underline{e}, \underline{f}_y(u) \right\rangle \mu_y(du) \right] \\ &+ (\alpha-\beta) p_2 \int_{\mathbf{E}} S^*(y,u,\Pi^1(y,u,\alpha,\beta), \Pi^2(y,u,\alpha,\beta)) \left\langle \underline{e}, \underline{f}_y(u) \right\rangle \mu_y(du) \Big\} \end{aligned}$$

Where:

$$R^*(y,z,\alpha,\beta) = \lim_{k\to\infty} R^k(y,z,\alpha,\beta)\ ,\quad S^*(y,z,\alpha,\beta) = \lim_{k\to\infty} S^k(y,z,\alpha,\beta)$$

Functions R^k and S^k are defined recursively:

$$R^1(y,z,\alpha,\beta) = 0\ ,\quad S^1(y,z,\alpha,\beta) = 1$$

$$\begin{aligned} R^{k+1}(y,z,\alpha,\beta) &= (1 - \mathbb{I}_{\mathcal{R}_k}(y,z,\alpha,\beta)) \\ &\times \left(p_1 \int_{\mathbf{E}} R^k(y,u,\Pi^1(y,u,\alpha,\beta), \Pi^2(y,u,\alpha,\beta))\mu_y(du) \right. \\ &\left. + q_1 \int_{\mathbf{E}} S^k(y,u,\Pi^1(y,u,\alpha,\beta), \Pi^2(y,u,\alpha,\beta)) \left\langle \underline{e}, \underline{f}_y(u) \right\rangle \mu_y(du) \right), \end{aligned} \tag{14}$$

$$\begin{aligned} S^{k+1}(y,z,\alpha,\beta) &= \mathbb{I}_{\mathcal{R}_k}(y,z,\alpha,\beta) + (1 - \mathbb{I}_{\mathcal{R}_k}(y,z,\alpha,\beta)) \\ &\times p_2 \int_{\mathbf{E}} S^k(y,u,\Pi^1(y,u,\alpha,\beta), \Pi^2(y,u,\alpha,\beta)) \left\langle \underline{e}, \underline{f}_y(u) \right\rangle \mu_y(du) \end{aligned} \tag{15}$$

Where the set $\mathcal{R}_k$ is:

$$
\begin{aligned}
\mathcal{R}_k = & \left\{(y,z,\alpha,\beta) : h(y,z,\alpha,\beta) \geq T_x Q_x^{k-1} h(y,z,\alpha,\beta)\right\} \\
= & \{(y,z,\alpha,\beta) : (\alpha-\beta) \geq (1-\alpha) \\
& \times \left[p_1 \int_{\mathbf{E}} R^k(y,u,\Pi^1(y,u,\alpha,\beta),\Pi^2(y,u,\alpha,\beta))\mu_y(du)\right. \\
& \left. + q_1 \int_{\mathbf{E}} S^k(y,u,\Pi^1(y,u,\alpha,\beta),\Pi^2(y,u,\alpha,\beta)) \left\langle \underline{e}, \underline{f}_y(u) \right\rangle \mu_y(du)\right] \\
& + (\alpha-\beta) p_2 \int_{\mathbf{E}} S^k(y,u,\Pi^1(y,u,\alpha,\beta),\Pi^2(y,u,\alpha,\beta)) \left\langle \underline{e}, \underline{f}_y(u) \right\rangle \mu_y(du) \Big\}
\end{aligned}
$$

(b) *The value problem. The optimal value for (10) is given by the formula*

$$
V(\tau^*) = p_1 \int_{\mathbf{E}} R^*(x,u,\varphi_x(u),0)\mu_x(du) + q_1 \int_{\mathbf{E}} S^*(x,u,\varphi_x(u),0) \left\langle \underline{e}, \underline{f}_x(u) \right\rangle \mu_x(du) \quad (16)
$$

where: $\varphi_x(u) = 1 - \frac{p_1}{p_1 + q_1 \langle \underline{e}, \underline{f}_x(u) \rangle}$

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