A detection of at least two disorders in random sequences

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Abstract. We register a random sequence constructed based on Markov processes by switching between them. At two random moments θ_1, θ_2, where $0 \leq \theta_1 \leq \theta_2$, the source of observations is changed. In effect the number of homogeneous segments is random. The transition probabilities of each process are known and *a priori* distribution of the disorder moments is given. Two cases are presented in details. In the first one the objective is to stop on between the disorder moments and in the second one our objective is to find the strategy which immediately detects the distribution changes. Both problems are reformulated to optimal stopping of the observed sequences. The detailed analysis of the problem is presented to show the form of optimal decision function.
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1 Introduction

Suppose that process $X = \{X_n, n \in \mathbb{N}\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, is observed sequentially. The process is obtained from three independent Markov processes by switching between them at two random moments of time, θ_1 and θ_2, independent of the switched processes. Our objective is to detect immediately these moments based on observation of X.

Shiryaev (1978) has considered the disorder problem for *independent random variables* with one disorder where the mean distance between disorder time and the moment of its detection was minimized. The probability maximizing approach to the problem was used by Bojdecki (1979) and the stopping time which is in a given neighborhood of the moment of disorder with maximal probability was found. The problem with two disorders was considered by Yoshida (1983) (detection of each disorders separately), Szajowski (1992) (sampling of the observation between disorders), Szajowski (1996) (detection of both disorders). Sarnowski and Szajowski (2008) have extended the results concerning sampling of the observations between disorders to the case with unknown distribution between disordered (see Bojdecki and Hosza (1984)). The methods of solution is based on reformulation of the question to the double optimal stopping problem (see Haggstrom (1967), Nikolaev (1979)) for markovian function of some statistics. The considerations has been inspired by the problem regarding how can we protect ourselves against a second fault in a technological system after the occurrence of an initial fault or by the problem of detection at the beginning and the end of an epidemic.

This paper is devoted to a generalization of the double disorder problem considered both in Szajowski (1992) and Szajowski (1996) in which immediate switch from the first preliminary distribution to the third one is possible with a positive probability. It is also possible that we observe the homogeneous data without disorders when both disorder moments are equal to 0. The extension leads to serious difficulties in the construction of equivalent double optimal stopping models.

2 Formulation of detection problems

Let $(X_n, n \in \mathbb{N})$ be an observable sequence of random variables defined on the space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $(\mathbf{E}, \mathcal{B})$, where $\mathbf{E}$ is a subset of $\mathbf{R}$. On $(\mathbf{E}, \mathcal{B})$ there are σ-additive measures $\{\mu_x\}_{x \in \mathbf{E}}$. Space $(\Omega, \mathcal{F}, \mathbf{P})$ supports variables θ_1, θ_2. They are $\mathcal{F}$-measurable variables with values in $\mathbb{N}$ with distributions:

$$\mathbf{P}(\theta_1 = j) = \mathbb{I}_{\{j=0\}}(j)\pi + \mathbb{I}_{\{j>0\}}(j)(1-\pi)p_1^{j-1}q_1,$$
$$\mathbf{P}(\theta_2 = k \mid \theta_1 = j) = \mathbb{I}_{\{k=j\}}(k)\rho + \mathbb{I}_{\{k>j\}}(k)(1-\rho)p_2^{k-j-1}q_2$$

where $j = 0, 1, 2, ...$, $k = j, j+1, j+2, ...$ Additionally we consider Markov processes $(X_n^i, \mathcal{G}_n^i, \mu_x^i)$ on $(\Omega, \mathcal{F}, \mathbf{P})$, $i = 0, 1, 2$, where σ-fields $\mathcal{G}_n^i$ are the smallest σ-fields for which (X^i), $i = 0, 1, 2$, are adapted, respectively. Let us define process $(X_n, n \in \mathbb{N})$ in the following way:

$$X_n = X_n^0 \cdot \mathbb{I}_{\{\theta_1 > n\}} + X_n^1 \cdot \mathbb{I}_{\{\theta_1 \le n < \theta_2\}} + X_n^2 \cdot \mathbb{I}_{\{\theta_2 \le n\}}.$$

We make inference based on the observable sequence $(X_n, n \in \mathbb{N})$ only. It should be emphasized that the sequence $(X_n, n \in \mathbb{N})$ is not markovian. However, the sequence satisfies the Markov property given θ_1 and θ_2 (see Szajowski (1996) and Moustakides (1998)). Let $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$, $n \in \mathbb{N}$. Measures $\mu_x^\bullet$ satisfy the relations: $\mu_x^i(dy) = f_x^i(y)\mu_x(dy)$, $i = 0, 1, 2$, where the functions $f_x^i(.)$ are different and $f_x^i(y)/f_x^{(i+1)\mathrm{mod}3}(y) < \infty$ for $i = 0, 1, 2$ and all $x, y \in \mathbf{E}$. We assume that the measures μ_x^i, $x \in \mathbf{E}$ are known in advance and we have that $\mathbf{P}(X_1^i \in A \mid X_0^i = x) = \int_A f_x^i(y)\mu_x(dy) = \mu_x^i(A)$ for every $A \in \mathcal{B}$ and $i \in \{0, 1, 2\}$.

Let $\mathcal{S}$ denote the set of all stopping times with respect to the filtration $(\mathcal{F}_n)$, $n = 0, 1, \ldots$ and $\mathcal{T} = \{(\tau, \sigma) : \tau \le \sigma,\ \tau, \sigma \in \mathcal{S}\}$. Two problems with three distributional segments are recalled to investigate them under weaker assumption that there are at most three homogeneous segments.

2.1 Detection of change

Our aim is to stop the observed sequence between the two disorders. We are looking for the stopping time $\tau^* \in \mathcal{S}$ such that

$$\mathbf{P}_x(\tau < \infty, \theta_1 \le \tau^* < \theta_2) = \sup_{\tau \in \mathcal{T}} \mathbf{P}_x(\tau < \infty, \theta_1 \le \tau < \theta_2).$$

2.2 Disorders detection

Our aim is to indicate the moments of switching with given precision d_1, d_2 (Problem $\mathrm{D}_{d_1 d_2}$). We want to determine a pair of stopping times $(\tau^*, \sigma^*) \in \mathcal{T}$ such that for every $x \in \mathbb{E}$

$$\mathbf{P}_x(|\tau^* - \theta_1| \le d_1, |\sigma^* - \theta_2| \le d_2) = \sup_{\substack{(\tau,\sigma)\in\mathcal{T} \\ 0 \le \tau \le \sigma < \infty}} \mathbf{P}_x(|\tau - \theta_1| \le d_1, |\sigma - \theta_2| \le d_2). \quad (1)$$

The problem has been considered in Szajowski (1996) under natural simplification that there are three segments of data (*i.e.* there is $0 < \theta_1 < \theta_2$). The solution of D_{00} problem will be presented in details.

3 On some *a posteriori* processes

The formulated problems will be translated to the optimal stopping problems for some Markov processes. The important part of the reformulation process is choice of the *statistics* describing knowledge of the decision maker. The *a posteriori* probabilities of some events play the crucial role. Let us define following *a posteriori* processes (cf. Yoshida (1983), Szajowski (1992)).

$$\Pi_n^i = \mathbf{P}_x(\theta_i \le n | \mathcal{F}_n), \quad (2)$$
$$\Pi_n^{12} = \mathbf{P}_x(\theta_1 = \theta_2 > n | \mathcal{F}_n) = P_x(\theta_1 = \theta_2 > n | \mathcal{F}_{mn}), \quad (3)$$
$$\Pi_{mn} = \mathbf{P}_x(\theta_1 = m, \theta_n > n | \mathcal{F}_{mn}), \quad (4)$$

for $m, n = 1, 2, \ldots$, $m < n$, $i = 1, 2$. For recursive representation of (2)–(4) we need following functions:

$$\Pi^1(x, y, \alpha, \beta, \gamma) = 1 - [p_1(1-\alpha)f_x^0(y)]\mathbf{H}^{-1}(x, y, \alpha, \beta, \gamma)$$
$$\Pi^2(x, y, \alpha, \beta, \gamma) = [(q_2\alpha + p_2\beta + q_1\gamma)f_x^2(y)]\mathbf{H}^{-1}(x, y, \alpha, \beta, \gamma)$$
$$\Pi^{12}(x, y, \alpha, \beta, \gamma) = p_1\gamma f_x^0(y)\mathbf{H}^{-1}(x, y, \alpha, \beta, \gamma)$$
$$\Pi(x, y, \alpha, \beta, \gamma, \delta) = p_2\delta f_x^1(y)\mathbf{H}^{-1}(x, y, \alpha, \beta, \gamma)$$

where $\mathbf{H}(x, y, \alpha, \beta, \gamma) = (1-\alpha)p_1 f_x^0(y) + [p_2(\alpha - \beta) + q_1(1 - \alpha - \gamma)]f_x^1(y) + [q_2\alpha + p_2\beta + q_1\gamma]f_x^2(y)$. In the sequel we adopt the following denotations: $\overrightarrow{\boldsymbol{\alpha}} = (\alpha, \beta, \gamma)$ and $\overrightarrow{\boldsymbol{\Pi}}_n = (\Pi_n^1, \Pi_n^2, \Pi_n^{12})$. The basic formulae used in the transformation of the disorder problems to the stopping problems are given in the following

Lemma 1. *For each $x \in \mathbb{E}$ and each Borel function $u : \Re \to \Re$ the following formulae for $m, n = 1, 2, \ldots$, $m < n$, $i = 1, 2$, hold:*

$$\Pi^i_{n+1} = \Pi^i(X_n, X_{n+1}, \Pi^1_n, \Pi^2_n, \Pi^{12}_n) \quad (5)$$
$$\Pi^{12}_{n+1} = \Pi^{12}(X_n, X_{n+1}, \Pi^1_n, \Pi^2_n, \Pi^{12}_n) \quad (6)$$
$$\Pi_{m\,n+1} = \Pi(X_n, X_{n+1}, \Pi^1_n, \Pi^2_n, \Pi^{12}_n, \Pi_{m\,n}) \quad (7)$$

with boundary condition $\Pi^1_0 = \pi$, $\Pi^2_0(x) = \pi\rho$, $\Pi_{m\,m} = (1-\rho)\frac{q_1 f^1_{X_{m-1}}(X_m)}{p_1 f^0_{X_{m-1}}(X_m)}(1 - \Pi^1_m)$.

Lemma 2. *For the model discribed in the section 2 the following formulae are valied.*

1. $\mathbf{P}_x(\theta_2 = \theta_1 > n+1|\mathcal{F}_n) = p_1\Pi^{12}_n$;
2. $\mathbf{P}_x(\theta_2 > \theta_1 > n+1|\mathcal{F}_n) = p_1(1 - \Pi^1_n - \Pi^{12}_n)$;
3. $\mathbf{P}_x(\theta_1 \le n+1|\mathcal{F}_n) = \mathbf{P}(\theta_1 \le n+1 < \theta_2|\mathcal{F}_n) + \mathbf{P}(\theta_2 \le n+1|\mathcal{F}_n)$;
4. $\mathbf{P}(\theta_1 \le n+1 < \theta_2|\mathcal{F}_n) = q_1(1 - \Pi^1_n - \Pi^{12}_n) + p_2(\Pi^1_n - \Pi^2_n)$;
5. $\mathbf{P}_x(\theta_{\le} n+1|\mathcal{F}_n) = q_2\Pi^1_n + p_2\Pi^2_n + q_1\Pi^{12}_n$.

Lemma 3. *For each $x \in \mathbf{E}$ and each Borel function $u : \mathbf{R} \longrightarrow \mathbf{R}$ the following equations are fulfilled*

$$\mathbf{E}_x\left(u(X_{n+1})(1 - \Pi^1_{n+1}) \mid \mathcal{F}_n\right) = (1 - \Pi^1_n - \Pi^{12}_n)p_1 \int_{\mathbb{E}} u(y) f^0_{X_n}(y)\mu_{X_n}(dy), \quad (8)$$
$$\mathbf{E}_x\left(u(X_{n+1})(\Pi^1_{n+1} - \Pi^2_{n+1}) \mid \mathcal{F}_n\right) = \left[q_1(1 - \Pi^1_n - \Pi^{12}_n) + p_2(\Pi^1_n - \Pi^2_n)\right] \int_{\mathbb{E}} u(x) f^1_{X_n}(y)\mu_{X_n}(dy), \quad (9)$$
$$\mathbf{E}_x\left(u(X_{n+1})\Pi^2_{n+1}) \mid \mathcal{F}_n\right) = \left[q_2\Pi^1_n + p_2\Pi^2_n + q_1\Pi^{12}_n\right] \int_{\mathbb{E}} u(y) f^2_{X_n}(y)\mu_{X_n}(dy), \quad (10)$$
$$\mathbf{E}_x\left(u(X_{n+1})\Pi^{12}_{n+1}) \mid \mathcal{F}_n\right) = \left[p_1\Pi^{12}_n\right] \int_{\mathbb{E}} u(y) f^0_{X_n}(y)\mu_{X_n}(dy) \quad (11)$$
$$\mathbf{E}_x(u(X_{n+1})|\mathcal{F}_n) = \int_{\mathbb{E}} u(y)\mathbf{H}(X_n, y, \overrightarrow{\boldsymbol{\Pi}}_n(x))\mu_{X_n}(dy) \quad (12)$$

4 Detection of new homogeneous segment

For $X_0 = x$ let us define: $Z_n = \mathbf{P}_x(\theta_1 \le n < \theta_2 \mid \mathcal{F}_n)$ for $n = 0, 1, 2, \ldots$. We have

$$Z_n = \mathbf{P}_x(\theta_1 \le n < \theta_2 \mid \mathcal{F}_n) = \Pi^1_n - \Pi^2_n \quad (13)$$

$Y_n = \mathrm{esssup}_{\{\tau \in \mathcal{T},\, \tau \ge \mathrm{n}\}} \mathbf{P}_{\mathrm{x}}(\theta_1 \le \tau < \theta_2 \mid \mathcal{F}_{\mathrm{n}})$ for $n = 0, 1, 2, \ldots$ and

$$\tau_0 = \inf\{n : Z_n = Y_n\} \quad (14)$$

Lemma 4. *The stopping time τ_0 defined by formula (14) is the solution of problem (2.1).*

The reduction of the disorder problem to optimal stopping of Markov sequence is the consequence of the following lemma.

Lemma 5. *System $X^x = \{X^x_n\}$, where $X^x_n = (X_{n-1}, X_n, \Pi^1_n, \Pi^2_n, \Pi^{12}_n)$ forms a family of random Markov functions.*

This fact implies that we can reduce initial problem (2.1) to the problem of optimal stopping five-dimensional process $(X_{n-1}, X_n, \Pi^1_n, \Pi^2_n, \Pi^{12}_n)$ with reward

$$h(x_1, x_2, \overrightarrow{\boldsymbol{\alpha}}) = \alpha - \beta \quad (15)$$

The reward function results from equation (13). Thanks to Lemma 5 we construct the solution using standard tools of optimal stopping theory (cf Shiryaev (1978)), as we do below.

For any Borel function $v:\mathbf{E}^2\times[0,1]^3\longrightarrow[0,1]$ and the set $D=\{\omega: X_{n-1}=y, X_n=z, \Pi_n^1=\alpha, \Pi_n^2=\beta, \Pi_n^{12}=\gamma\}$ let us define two operators:

$$T_x v(y,z,\overrightarrow{\boldsymbol{\alpha}})=E_x(v(X_n,X_{n+1},\overrightarrow{\boldsymbol{\Pi}}_{n+1})\mid D)$$
$$\mathbf{Q}_x v(y,z,\overrightarrow{\boldsymbol{\alpha}})=\max\{v(y,z,\overrightarrow{\boldsymbol{\alpha}}),\mathbf{T}_x v(y,z,\overrightarrow{\boldsymbol{\alpha}})\}$$

From well known theorems of optimal stopping theory (Shiryaev (1978)), we infer that the solution of the problem (2.1) is the Markov time τ_0:

$$\tau_0=\inf\{h(X_n,X_{n+1},\overrightarrow{\boldsymbol{\Pi}}_{n+1})\geq h^*(X_n,X_{n+1},\overrightarrow{\boldsymbol{\Pi}}_{n+1})\}$$

where $h^*(y,z,\overrightarrow{\boldsymbol{\alpha}})=\lim_{k\to\infty}\mathbf{Q}_x^k h(y,z,\overrightarrow{\boldsymbol{\alpha}})$. Of course $\mathbf{Q}_x^k v(y,z,\overrightarrow{\boldsymbol{\alpha}})=\max\{\mathbf{Q}_x^{k-1}v,\mathbf{T}_x\mathbf{Q}_x^{k-1}v\}=\max\{v,\mathbf{T}_x\mathbf{Q}_x^{k-1}v\}$. To obtain a clearer formula for τ_0, we formulate

Theorem 1. **(a)** *The solution of problem (2.1) is given by:*

$$\tau^*=\inf\{n:(X_n,X_{n+1},\overrightarrow{\boldsymbol{\Pi}}_{n+1})\in B^*\} \tag{16}$$

Set B^* is of the form:

$$B^*=\Big\{(y,z,\overrightarrow{\boldsymbol{\alpha}}):(\alpha-\beta)\geq(1-\alpha)\Big[p_1\int_{\mathbb{E}}R^*(y,u,\overrightarrow{\boldsymbol{\Pi}}_1(y,u,\overrightarrow{\boldsymbol{\alpha}}))f_y^0(u)\mu_y(du)$$
$$+\,q_1\int_{\mathbb{E}}S^*(y,u,\overrightarrow{\boldsymbol{\Pi}}_1(y,u,\overrightarrow{\boldsymbol{\alpha}}))f_y^1(u)\mu_y(du)\Big]+(\alpha-\beta)p_2\int_{\mathbb{E}}S^*(y,u,\overrightarrow{\boldsymbol{\Pi}}_1(y,u,\overrightarrow{\boldsymbol{\alpha}}))f_y^1(u)\mu_y(du)\Big\}$$

Where $R^*(y,z,\overrightarrow{\boldsymbol{\alpha}})=\lim_{k\to\infty}R^k(y,z,\overrightarrow{\boldsymbol{\alpha}})$, $S^*(y,z,\overrightarrow{\boldsymbol{\alpha}})=\lim_{k\to\infty}S^k(y,z,\overrightarrow{\boldsymbol{\alpha}})$. Functions R^k and S^k are defined recursively.

(b) *The optimal value for (2.1) is given by the formula $V(\tau^*)=p_1\int_{\mathbb{E}}R^*(x,u,\overrightarrow{\boldsymbol{\Pi}}_1(x,u,\pi,\rho\pi,\rho(1-\pi)))f_x^0(u)\mu_x(du)+q_1\int_{\mathbb{E}}S^*(x,u,\overrightarrow{\boldsymbol{\Pi}}_1(x,u,\pi,\rho\pi,\rho(1-\pi)))f_x^1(u)\mu_x(du)$.*

It is notable that the solution of formulated problem depends only on two-dimensional vector of posterior processes because $\Pi_n^{12}=\rho(1-\Pi_n^1)$. The formulas obtained are very general and for this reason - quite complicated. We simplify the model by assuming that $P(\theta_1>0)=1$ and $P(\theta_2>\theta_1)=1$. However, it seems that some further simplifications can be made in special cases. Further research should be carried out in this direction. From a practical point of view, computer algorithms are necessary to construct B^* – the set in which it is optimally to stop our observable sequence.

5 Immediate detection of the first and the second disorder

5.1 Equivalent double optimal stopping problem

Let us consider the problem formulated in (1). A *compound stopping variable* is a pair (τ,σ) of stopping times such that $\tau\leq\sigma$ a.e.. Denote $\mathcal{T}_m=\{(\tau,\sigma)\in\mathcal{T}:\tau\geq m\}$, $\mathcal{T}_{mn}=\{(\tau,\sigma)\in\mathcal{T}:\tau=m,\sigma\geq n\}$ and $\mathcal{S}_m=\{\tau\in\mathcal{S}:\tau\geq m\}$. Let us denote $\mathcal{F}_{mn}=\mathcal{F}_n$, $m,n\in\mathbb{N}$, $m\leq n$. We define two-parameter stochastic sequence $\xi(x)=\{\xi_{mn},\ m,n\in\mathbb{N},\ m<n,\ x\in\mathbb{E}\}$, where $\xi_{mn}=\mathbf{P}_x(\theta_1=m,\theta_2=n|\mathcal{F}_{mn})$.

We can consider for every $x\in\mathbb{E}$, $m,n\in\mathbb{N}$, $m<n$, the optimal stopping problem of $\xi(x)$ on $\mathcal{T}_{mn}^+=\{(\tau,\sigma)\in\mathcal{T}_{mn}:\tau<\sigma\}$. A compound stopping variable (τ^*,σ^*) is said to be optimal in $\mathcal{T}_m^+$ (or $\mathcal{T}_{mn}^+$) if $\mathbf{E}_x\xi_{\tau^*\sigma^*}=\sup_{(\tau,\sigma)\in\mathcal{T}_m^+}\mathbf{E}_x\xi_{\tau\sigma}$ (or $\mathbf{E}_x\xi_{\tau^*\sigma^*}=\sup_{(\tau,\sigma)\in\mathcal{T}_{mn}^+}\mathbf{E}_x\xi_{\tau\sigma}$). Let us define

$$\eta_{mn}=\operatorname*{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}_{mn}^+}\mathbf{E}_x(\xi_{\tau\sigma}|\mathcal{F}_{mn}). \tag{17}$$

If we put $\xi_{m\infty}=0$, then $\eta_{mn}=\operatorname{ess\,sup}_{(\tau,\sigma)\in\mathcal{T}_{mn}^+}\mathbf{P}_x(\theta_1=\tau,\theta_2=\sigma|\mathcal{F}_{mn})$. From the theory of optimal stopping for double indexed processes (cf. Haggstrom (1967),Nikolaev (1981)) the sequence η_{mn} satisfies $\eta_{mn}=\max\{\xi_{mn},\mathbf{E}(\eta_{mn+1}|\mathcal{F}_{mn})\}$. Moreover, if $\sigma_m^*=\inf\{n>m:\eta_{mn}=\xi_{mn}\}$, then (m,σ_n^*) is optimal in $\mathcal{T}_{mn}^+$ and $\eta_{mn}=\mathbf{E}_x(\xi_{m\sigma_n^*}|\mathcal{F}_{mn})$ a.e.. Define $\hat{\eta}_{mn}=\max\{\xi_{mn},\mathbf{E}(\eta_{m\,n+1}|\mathcal{F}_{mn})\}$ for $n\geq m$ if $\hat{\sigma}_m^*=\inf\{n\geq m:\hat{\eta}_{mn}=\xi_{mn}\}$, then $(m,\hat{\sigma}_m^*)$ is optimal in $\mathcal{T}_{mn}$ and $\hat{\eta}_{mm}=\mathbf{E}_x(\xi_{m\sigma_m^*}|\mathcal{F}_{mm})$ a.e.. For further consideration denote $\eta_m=\mathbf{E}_x(\eta_{mm+1}|\mathcal{F}_m)$.

Lemma 6. *Stopping time σ_m^* is optimal for every stopping problem (17).*

What is left is to consider the optimal stopping problem for $(\eta_{mn})_{m=0,n=m}^{\infty,\infty}$ on $(\mathcal{T}_{mn})_{m=0,n=m}^{\infty,\infty}$. Let us define

$$V_m = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_m} \mathbf{E}_x(\eta_\tau|\mathcal{F}_m). \tag{18}$$

Then $V_m = \max\{\eta_m, \mathbf{E}_x(V_{m+1}|\mathcal{F}_m)\}$ a.e. and we define $\tau_n^* = \inf\{k \geq n : V_k = \eta_k\}$.

Lemma 7. *The strategy τ_0^* is the optimal strategy of the first stop.*

Lemmas 6 and 7 describe the method of solving the *disorder problem* formulated in Section 2 (see (1)).

5.2 Solution of the equivalent double stopping problem

For the sake of simplicity we shall confine ourselves to the case $d_1 = d_2 = 0$. It will be easily seen how to generalize the solution of the problem to solve $D_{d_1 d_2}$ for $d_1 > 0$ or $d_2 > 0$. First of all we construct multidimensional Markov chains such that ξ_{mn} and η_m will be the functions of their states. By consideration of the section 3 concerning *a posteriori* processes we get $\xi_{00} = \pi\rho$ and for $m < n$

$$\xi_{m\,n}^x \stackrel{L.2}{=} \mathbf{P}_x(\theta_1 = m, \theta_2 = n|\mathcal{F}_{m\,n}) = \begin{cases} \frac{q_2}{p_2}\Pi_{m\,n}(x)\frac{f^2_{X_{n-1}}(X_n)}{f^1_{X_{n-1}}(X_n)} & \text{for } m < n \\ \rho\frac{q_1}{p_1}\frac{f^2_{X_{m-1}}(X_m)}{f^0_{X_{m-1}}(X_m)}(1-\Pi^1_m) & \text{for } n = m. \end{cases} \tag{19}$$

We can observe that $(X_n, X_{n+1}, \vec{\boldsymbol{\Pi}}_n, \Pi_{m\,n})$ for $n = m+1, m+2, \ldots$ is a function of $(X_{n-1}, X_n, \vec{\boldsymbol{\Pi}}_{n-1}, \Pi_{m\,n-1})$ and X_{n+1}. Besides the conditional distribution of X_{n+1} given $\mathcal{F}_n$ (cf. (12)) depends on X_n, $\Pi_n^1(x)$ and $\Pi_n^2(x)$ only. These facts imply that $\{(X_n, X_{n+1}, \vec{\boldsymbol{\Pi}}_n, \Pi_{m\,n})\}_{n=m+1}^{\infty}$ form a homogeneous Markov process (see Chapter 2.15 of Shiryaev (1978)). This allows us to reduce the problem (17) for each m to the optimal stopping problem of the Markov process $Z_m(x) = \{(X_{n-1}, X_n, \vec{\boldsymbol{\Pi}}_n, \Pi_{m\,n}),\ m, n \in \mathbb{N},\ m < n,\ x \in \mathbb{E}\}$ with the reward function $h(t, u, \vec{\boldsymbol{\alpha}}, \delta) = \frac{q_2}{p_2}\delta\frac{f_t^2(u)}{f_t^1(u)}$.

Lemma 8. *A solution of the optimal stopping problem (17) for $m = 1, 2, \ldots$ has a form*

$$\sigma_m^* = \inf\{n > m : \frac{f^2_{X_{n-1}}(X_n)}{f^1_{X_{n-1}}(X_n)} \geq R^*(X_n)\}$$

where $R^*(t) = p_2\int_{\mathbb{E}} r^*(t,s)f_t^1(s)\mu_t(ds)$ and the function $r^*(t,u)$ satisfies the equation $r^*(t,u) = \max\{\frac{f_t^2(u)}{f_t^1(u)}, p_2\int_{\mathbb{E}} r^*(u,s)f_u^1(s)\mu_u(ds)\}$. The value of the problem is equal

$$\eta_m = \mathbf{E}_x(\eta_{m\,m+1}|\mathcal{F}_m) = \frac{q_1}{p_1}\frac{f^1_{X_{m-1}}(X_m)}{f^0_{X_{m-1}}(X_m)}(1-\Pi_m^1)R_\rho^\star(X_{m-1}, X_m),$$

where $R_\rho^\star(t,u) = \max\{\rho\frac{f_t^2(u)}{f_t^1(u)}, \frac{q_2}{p_2}(1-\rho)R^\star(t,u)\}$.

Based on the results of Lemma 8 and properties of the *a posteriori* process Π_{nm} we have optimal second moment

$$\hat{\sigma}_0^\star = \begin{cases} 0 & \text{if } \pi\rho \geq q_1(1-\pi)\int_{\mathbb{E}} f_x^1(u)R_\rho^\star(x,u)\mu_x(du), \\ \sigma_0^\star & \text{otherwise.} \end{cases}$$

By lemmas 8 and 1 (formula (7)) the optimal stopping problem (18) has been transformed to the optimal stopping problem for the homogeneous Markov process $W = \{(X_{m-1}, X_m, \vec{\boldsymbol{\Pi}}_m, \Pi_m^{12}),\ m \in \mathbb{N},\ x \in \mathbb{E}\}$ with the reward function

$$f(t, u, \vec{\boldsymbol{\alpha}}) = \frac{q_1}{p_1}\frac{f_t^1(u)}{f_t^0(u)}(1-\alpha)R_\rho^\star(t,u).$$

Theorem 2. *A solution of the optimal stopping problem* (18) *for* $n = 1, 2, \ldots$ *has a form*

$$\tau_n^* = \inf\{k \geq n : (X_{k-1}, X_k, \overrightarrow{\boldsymbol{\Pi}}_k,) \in B^*\} \tag{20}$$

where $B^* = \{(t, u, \overrightarrow{\boldsymbol{\alpha}}) : \frac{f_t^2(u)}{f_t^1(u)} R_\rho^\star(t, u) \geq p_1 \int_{\mathbb{E}} v^*(u, s) f_u^0(s) \mu_u(ds)\}$. *The function* $v^*(t, u) = \lim_{n\to\infty} v_n(t, u)$, *where* $v_0(t, u) = R_\rho^\star(t, u)$,

$$v_{n+1}(t, u) = \max\{\frac{f_t^2(u)}{f_t^1(u)} R_\rho^\star(t, u), p_1 \int_{\mathbb{E}} v_n(u, s) f_u^1(s) \mu_u(ds)\}. \tag{21}$$

So $v^*(t, u)$ *satisfies the equation* $v^*(t, u) = \max\{\frac{f_t^2(u)}{f_t^1(u)} R_\rho^\star(t, u), p_1 \int_{\mathbb{E}} v^*(u, s) f_u^1(s) \mu_u(ds)\}$. *The value of the problem* $V_n = v^*(X_{n-1}, X_n)$.

Based on Lemmas 8 and 2 the solution of the problem D_{00} can be formulated as follows.

Theorem 3. *A compound stopping time* $(\tau^*, \sigma_{\tau^*}^*)$, *where* σ_m^* *is given by* (8) *and* $\tau^* = \hat{\tau}_0^*$ *is given by* (20) *is a solution of the problem* D_{00}. *The value of the problem*

$$\mathbf{P}_x(\tau^* < \sigma^* < \infty, \theta_1 = \tau^*, \theta_2 = \sigma_{\tau^*}^*) = \max\{\pi, q_1(1 - \pi) \int_{\mathbb{E}} v^*(u, s) f_u^0(s) \mu_u(ds)\}.$$

Remark 1. The problem can be extended to optimal detection of more than two successive disorders. The distribution of θ_1, θ_2 may be more general. The general *a priori* distributions of disorder moments leads to more complicated formulae, since the corresponding Markov chains are not homogeneous.

6 Final remarks

The formulated problems are translated to the optimal stopping problems for some multidimensional Markov processes. The important part of the reformulation process is choice of the *statistics* describing knowledge of the decision maker. The *a posteriori* probabilities of some events play the crucial role. The optimal stopping times have been constructed and the optimal value of the problems has been calculated for both problems (see Szajowski (2009)). The extension of the models when the knowledge about densities in each segment is limited to the information about sets of possible conditional densities is included to Sarnowski and Szajowski (2009).

Bibliography

Bojdecki, T., 1979. Probability maximizing approach to optimal stopping and its application to a disorder problem. Stochastics 3, 61–71.

Bojdecki, T., Hosza, J., 1984. On a generalized disorder problem. Stochastic Processes Appl. 18, 349–359.

Haggstrom, G., 1967. Optimal sequential procedures when more then one stop is required. Ann. Math. Statist. 38, 1618–1626.

Moustakides, G., 1998. Quickest detection of abrupt changes for a class of random processes. IEEE Trans. Inf. Theory 44 (5), 1965–1968.

Nikolaev, M., 1979. Obobshchennye posledovatel′nye procedury. Litovskiĭ Matematicheskiĭ Sbornik 19, 35–44.

Nikolaev, M., 1981. O kriterii optimal'nosti obobshchennoi posledovatel'noj procedury. Litov. Mat. Sb. 21, 75–82, On the criterion of optimality of the extended sequential procedure (in Russian).

Sarnowski, W., Szajowski, K., 2008. On-line detection of a part of a sequence with unspecified distribution. Stat. Probab. Lett. 78 (15), 2511–2516, doi:10.1016/j.spl.2008.02.040.

Sarnowski, W., Szajowski, K., 2009. Unspecified distributions in disorder problem. In: Mukhopadhyay, N., Moustakides, G. V. (Eds.), Extended Abstracts of the Second International Workshop in Sequential Methodology, Troyes 15-17.06.2009. University of Technology of Troyes, France, Troyes, pp. IWSM Article No.: 81, 6pages.

Shiryaev, A., 1978. Optimal Stopping Rules. Springer-Verlag, New York, Heidelberg, Berlin.

Szajowski, K., 1992. Optimal on-line detection of outside observation. J.Stat. Planning and Inference 30, 413–426.

Szajowski, K., 1996. A two-disorder detection problem. Appl. Math. 24 (2), 231–241.

Szajowski, K., January 2009. On a random number of disorders. Tech. rep., Institute of Mathematics, PAS, Wrocław, Poland, arxiv.org/abs/0901.3795v1.

Yoshida, M., 1983. Probability maximizing approach for a quickest detection problem with complicated Markov chain. J. Inform. Optimization Sci. 4, 127–145.