On asymptotic normality of sequential LS-estimates of unstable autoregressive processes.

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Abstract. Sequential least squares estimates are proposed for estimating the unknown parameters in an unstable autoregressive $AR(p)$. A special stopping time is defined by the trace of the observed Fisher information matrix. In the case $p = 2$, the limiting distribution of the sequential LSE is shown to be normal for the parameter vector lying both inside the stability region and on its boundary in contrast to the usual LSE. In the case $p \geq 3$, the asymptotic normality of the sequential LSE is shown for the parameter vector lying both inside the stability region and on some part of its boundary. This asymptotic normality is provided by a new property of the observed Fisher information matrix which holds both inside the stability region and on some part of its boundary.

Keywords. Autoregressive process, least squares estimate, sequential estimate, asymptotic normality.

1 Introduction

Consider the autoregressive $AR(p)$ model

$$x_n = \theta_1 x_{n-1} + \ldots + \theta_p x_{n-p} + \varepsilon_n,\ n = 1, 2, \ldots, \tag{1}$$

where (x_n) is the observation, (ε_n) is the noise which is a sequence of independent identically distributed (i.i.d.) random variables with $E\varepsilon_1 = 0$ and $0 < E\varepsilon_1^2 = \sigma^2 < \infty$, σ^2 is known (or unknown), $x_0 = x_{-1} = \ldots = x_{1-p} = 0$; the parameters $\theta_1, \ldots, \theta_p$ of the model are unknown.

A commonly used estimate of the parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ is the least squares estimate (LSE)

$$\boldsymbol{\theta}(n) = M_n^{-1} \sum_{k=1}^{n} X_{k-1} x_k, \quad M_n = \sum_{k=1}^{n} X_{k-1} X_{k-1}', \tag{2}$$

where $X_n = (x_n, x_{n-1}, \ldots, x_{n-p+1})'$, M_n^{-1} denotes the inverse of matrix M_n if $\det M_n > 0$ and $M_n^{-1} = 0$ otherwise, the prime denotes the transposition; M_n is called the observed Fisher information matrix. Let

$$\mathcal{P}(z) = z^p - \theta_1 z^{p-1} - \ldots - \theta_p \tag{3}$$

denote the characteristic polynomial of the autoregressive model (1). The process (1) is said to be stable if all roots $z_i = z_i(\boldsymbol{\theta})$ of the characteristic polynomial (3) lie inside the unit circle, that is the parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ belongs to the parametric stability region Λ_p defined as

$$\Lambda_p = \{\boldsymbol{\theta} \in R^p : |z_i(\boldsymbol{\theta})| < 1, i = 1, \ldots, p\}. \tag{4}$$

The process (1) is called unstable if the roots of $\mathcal{P}(z)$ lie on or inside the unit circle, that is, $\boldsymbol{\theta} \in [\Lambda_p]$, where $[\Lambda_p]$ denotes the closure of the stability region Λ_p.

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It is well known (see,e.g. Anderson (1971), Th.5.5.7) that the LSE $\boldsymbol{\theta}(n)$ is asymptotically normal for all $\boldsymbol{\theta} \in \Lambda_p$, that is

$$\sqrt{n}(\boldsymbol{\theta}(n) - \boldsymbol{\theta}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, F), \text{ as } n \to \infty,$$

where $F = F(\boldsymbol{\theta})$ is a positive definite matrix, $\xrightarrow{\mathcal{L}}$ indicates convergence in law. It should be noted that the asymptotic normality of $\boldsymbol{\theta}(n)$ is provided by the following asymptotic property of the observed Fisher information matrix

$$\lim_{n\to\infty} M_n/n = F \text{ a.s.} \tag{5}$$

for all $\boldsymbol{\theta} \in \Lambda_p$. On the boundary $\partial\Lambda_p$ of the stability region Λ_p, this property does not hold and the distribution of $\boldsymbol{\theta}(n)$ is no longer asymptotically normal.

The investigation of the asymptotic distribution of LSE $\boldsymbol{\theta}(n)$ when (x_n) is unstable goes back to the late fifties with the paper of White (1958) (see also Ahtola and Tiao (1987), Dickey and Fuller (1979), Rao (1978), Sriram (1987),(1988)) who considered the AR(1) model with i.i.d. $\mathcal{N}(0, \sigma^2)$ random errors ε_n and $\theta = 1$, and established that

$$n(\theta(n) - 1) \xrightarrow{\mathcal{L}} (W^2(1) - 1)/\int_0^1 W^2(t)dt\,,$$

where $W(t)$ is a standard brownian motion. Subsequently the research of the limiting distribution of $\boldsymbol{\theta}(n)$ for unstable AR(p) processes has been receiving considerable attention due to important applications in time series analysis, in modeling economic and financial data and in system identification and control. For the detail we refer the reader to the paper by Chan and Wei (1988) who derived the limiting distribution of LSE $\boldsymbol{\theta}(n)$ for the general unstable AR(p) model. By making use of the functional central limit theorem approach, Chan and Wei expressed the limiting distribution of LSE $\boldsymbol{\theta}(n)$ in terms of functionals of standard brownian motions. However, the closed forms of the distribution functions of these functionals are not known and that may cause difficulties in practice (see section 4 in Chan and Wei).

For the unstable AR(1) model with i.i.d. random errors and $-1 \le \theta \le 1$, Lai and Siegmund (1983) proposed, for θ, to use the sequential least squares estimate

$$\theta(\tau) = \left(\sum_{k=1}^{\tau} x_{k-1}^2\right)^{-1} \sum_{k=1}^{\tau} x_{k-1}x_k\,,\ \tau = \tau(h) = \inf\{n \ge 1 : \sum_{k=1}^{n} x_{k-1}^2 \ge h\sigma^2\}. \tag{6}$$

They proved that, in contrast with the ordinary LSE $\theta(n)$, the sequential LSE is asymptotically normal uniformly in $\theta \in [-1, 1]$.

In the next section, for the unstable AR(2) model, we apply the sequential LSE with a particular stopping time based on the trace of the observed Fisher information matrix and establish that it is asymptotically normal not only inside the stability region Λ but also on its boundary in contrast to the usual LSE (see Galtchouk and Konev (2008)). In section 3, for the case of unstable AR(p), $p \ge 3$, process, we propose a sequential LSE for $\boldsymbol{\theta}$ and find the conditions on $\boldsymbol{\theta}$ (see Conditions 1-3 hereafter) ensuring its asymptotic normality. The set $\tilde{\Lambda}_p$ of the points $\boldsymbol{\theta}$, satisfying these conditions includes the stability region Λ_p and some part of its boundary. It is shown that the convergence of the sequential LSE to the normal distribution is uniform in $\boldsymbol{\theta} \in K$ for any compact set $K \in \tilde{\Lambda}_p$.

2 Asymptotic normality of the sequential LSE for AR(2).

In this section we develop a sequential sampling scheme for estimating parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ in model (1) with $p = 2$. We will use the sequential least squares estimate defined by the formula

$$\boldsymbol{\theta}(\tau(h)) = M_{\tau(h)}^{-1} \sum_{k=1}^{\tau(h)} X_{k-1}x_k, \tag{7}$$

where $\tau(h)$ is the stopping time for the threshold $h > 0$:

$$\tau(h) = \inf\{n \geq 1 : trM_n \geq h\sigma^2\},\ \inf\{\emptyset\} = +\infty. \tag{8}$$

This construction of sequential estimate is similar to that in (6) of Lai and Siegmund for $AR(1)$. It should be noted, however, that the first factor on the right-hand side of (7) is a random matrix and not a random variable, as in (6), and this makes additional difficulties.

For AR(1) the stopping time τ in (6) turns the denominator in $\theta(\tau)$ (6) practically into a constant $h\sigma^2$ and this allows to use the central limit theorem for martingales. In the case of $AR(2)$ the stopping time (8) enables one to control the inverse matrix $M^{-1}_{\tau(h)}$ in (7) only partially since it remains random. Nevertheless, we will see that such a change of time also enables one to improve the properties of the estimate (2).

In our paper (2006) we proved the following result.

Theorem 1. *Let $(\varepsilon_n)_{n\geq 1}$ in (1) with $p = 2$ be a sequence of i.i.d. random variables with $E\varepsilon_n = 0,\ 0 < E\varepsilon_n^2 = \sigma^2 < \infty, \sigma^2$ is known. Then, for any compact set $K \subset \Lambda_2^*$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \sup_{\boldsymbol{t}\in R^2} |P_{\boldsymbol{\theta}} \left(M^{1/2}_{\tau(h)}(\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) \leq \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t}/\sigma)| = 0,$$

where $\Phi_2(\boldsymbol{t}) = \Phi(t_1)\Phi(t_2),\ \Phi$ is the standard normal distribution function,

$$\Lambda_2^* = \{\boldsymbol{\theta} = (\theta_1, \theta_2)' :\ -1 + \theta_2 < \theta_1 < 1 - \theta_2,\ -1 \leq \theta_2 < 1\},\ \boldsymbol{t} = (t_1, t_2)'.$$

This theorem implies, in particular, that estimate (7) is asymptotically normal uniformly on the compact sets not only inside the stability region (4) but also on the part of its boundary $\{\boldsymbol{\theta} = (\theta_1, -1)' : -2 < \theta_1 < 2\}$ in contrast to the LSE (2).

The following result claims that the asymptotic normality of the estimate (7),(8) holds in the whole region $[\Lambda_2]$ including its boundary $\partial\Lambda_2$.

Theorem 2. *Suppose that in model (1) with $p = 2$, $(\varepsilon_n)_{n\geq 1}$ is a sequence of i.i.d. random variables, $E\varepsilon_n = 0$ and $0 < E\varepsilon_n^2 = \sigma^2 < \infty, \sigma^2$ is known. Define $\tau(h),\ \boldsymbol{\theta}(\tau(h))$ and $M_{\tau(h)}$ as in (8),(7) and (2). Then for any $\boldsymbol{\theta} \in [\Lambda_2]$*

$$\lim_{h\to\infty} \sup_{\boldsymbol{t}\in R^2} \left| P_{\boldsymbol{\theta}} \left(M^{1/2}_{\tau(h)}(\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) \leq \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t}/\sigma) \right| = 0,$$

where $\Phi_2(\boldsymbol{t}) = \Phi(t_1)\Phi(t_2),\ \boldsymbol{t} = (t_1, t_2)',\ \Phi$ is the standard normal distribution function; $[\Lambda_2]$ is the closure of the stability region (4).

In the both previous results the variance σ^2 is supposed to be known.

Suppose now that the variance σ^2 in (1) is unknown. A commonly used estimate for σ^2 in autoregressive processes on the basis of observations $(x_1, \ldots, x_n)$ is defined as

$$\hat{\sigma}_n^2 = n^{-1} \sum_{k=1}^{n} (x_k - \boldsymbol{\theta}'(n) X_{k-1})^2 , \tag{9}$$

where $\boldsymbol{\theta}(n)$ is the least squares estimate of $\boldsymbol{\theta}$ defined in (2). Now we must modify the stopping time (8). At first sight, to this end one should replace σ^2 in (8) by $\hat{\sigma}_n^2$. However, we will use a different modification similar to that proposed by Lai and Siegmund for AR(1) model, which turns out to be more convenient in the theoretic studies. Define the sequential estimate as

$$\boldsymbol{\theta}(\hat{\tau}(h)) = M^{-1}_{\hat{\tau}(h)} \sum_{k=1}^{\hat{\tau}(h)} X_{k-1} x_k , \tag{10}$$

$$\hat{\tau}(h) = \inf\{n \geq 3 : \sum_{k=1}^{n} (x_{k-1}^2 + x_{k-2}^2) \geq h s_n^2\} , \tag{11}$$

where $s_n^2 = \hat{\sigma}_n^2 \vee \delta_n,\ \delta_n$ is a sequence of positive numbers with $\delta_n \to 0$.

The asymptotic normality in the case of unknown variance is stated in the following theorems.

Theorem 3. *Let $(\varepsilon_n)_{n\ge 1}$ in (1) with $p=2$ be a sequence of i.i.d. random variables, $E\varepsilon_n = 0,\ 0 < E\varepsilon_n^2 = \sigma^2 < \infty, \sigma^2$ is unknown. Then, for any compact set $K \subset \Lambda_2^*$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{\theta}\in K} \sup_{\boldsymbol{t}\in R^2} |P_{\boldsymbol{\theta}} \left(M_{\hat{\tau}(h)}^{1/2} (\boldsymbol{\theta}(\hat{\tau}(h)) - \boldsymbol{\theta})/\hat{\sigma}_{\hat{\tau}(h)} \le \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t})| = 0,$$

where $\Phi_2(\boldsymbol{t}) = \Phi(t_1)\Phi(t_2),\ \Phi$ is the standard normal distribution function,

$$\Lambda_2^* = \{\boldsymbol{\theta} = (\theta_1,\theta_2)' :\ -1+\theta_2 < \theta_1 < 1-\theta_2,\ -1 \le \theta_2 < 1\},\ \boldsymbol{t} = (t_1,t_2)' .$$

Theorem 4. *Let $(\varepsilon_n)_{n\ge 1}$ in (1) with $p=2$ be a sequence of i.i.d. random variables, $E\varepsilon_n = 0,\ 0 < E\varepsilon_n^2 = \sigma^2 < \infty, \sigma^2$ is unknown. Then, for any $\boldsymbol{\theta} \in [\Lambda_2]$,*

$$\lim_{h\to\infty} \sup_{\boldsymbol{t}\in R^2} |P_{\boldsymbol{\theta}} \left(M_{\hat{\tau}(h)}^{1/2} (\boldsymbol{\theta}(\hat{\tau}(h)) - \boldsymbol{\theta})/\hat{\sigma}_{\hat{\tau}(h)} \le \boldsymbol{t} \right) - \Phi_2(\boldsymbol{t})| = 0 .$$

Now we study the asymptotic behavior of the stopping time $\tau(h)$.

The boundary $\partial\Lambda_2$ includes three sides:

$$\Gamma_1 = \{\boldsymbol{\theta} : -\theta_1+\theta_2 = 1, -2<\theta_1<0\}\,, \Gamma_2 = \{\boldsymbol{\theta} : \theta_1+\theta_2 = 1, 0<\theta_1<2\}\,,$$
$$\Gamma_3 = \{\boldsymbol{\theta} : -2<\theta_1<2, \theta_2 = -1\} \tag{12}$$

and three apexes $(0,1), (-2,-1), (2,-1)$. Denote

$$A = \begin{pmatrix} \theta_1 & \theta_2 \\ 1 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$W^{(n)}(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=0}^{[nt]} \varepsilon_i,\ W_1^{(n)}(t) = \frac{1}{\sigma\sqrt{n}} \sum_{i=0}^{[nt]} (-1)^i \varepsilon_i,\ 0 \le t \le 1, \tag{13}$$

and introduce the following functionals

$$\mathcal{J}_1(x;t) = \int_0^t x^2(s)ds,\ \mathcal{J}_2(x;t) = \int_0^t \left(\int_0^s x(u)du\right)^2 ds, \tag{14}$$

$$\mathcal{J}_3(x;y;t) = \int_0^t (x^2(s)+y^2(s))ds,\ \mathcal{J}_4(x;t) = \left(\int_0^t x(s)ds\right)^2 .$$

Theorem 5. *Let $(\varepsilon_n)_{n\ge 1}$ in (1.1) be a sequence of i.i.d. random variables with $E\varepsilon_n = 0, E\varepsilon_n^2 = \sigma^2$ and $\tau(h)$ be defined by (8). Denote by a and b real roots of the polynomial (3) with $p=2,\ -1 \le a < b \le 1$. Then, for each $\boldsymbol{\theta} \in \Lambda_2$,*

$$P_{\boldsymbol{\theta}} - \lim_{h\to\infty} \tau(h)/h = 1/trF,\ F - AFA' = B. \tag{15}$$

Moreover, for each $\boldsymbol{\theta} \in \partial\Lambda_2$, as $h\to\infty$,

$$\frac{\tau(h)}{\psi(\boldsymbol{\theta},h)} \stackrel{\mathcal{L}}{\Longrightarrow} \begin{cases} \nu_1(W_1) = \inf\{t\ge 0 : \mathcal{J}_1(W_1;t)\ge 1\} \text{ if } \boldsymbol{\theta}\in\Gamma_1, \\ \nu_2(W) = \inf\{t\ge 0 : \mathcal{J}_1(W;t)\ge 1\} \text{ if } \boldsymbol{\theta}\in\Gamma_2, \\ \nu_3(W,W_1) = \inf\{t\ge 0 : \mathcal{J}_3(W;W_1;t)\ge 1\} \text{ if } \boldsymbol{\theta}\in\Gamma_3\cup\{(0,1)\}, \\ \nu_4(W) = \inf\{t\ge 0 : \mathcal{J}_2(W;t)\ge 1\} \text{ if } \boldsymbol{\theta} = (2,-1), \\ \nu_5(W_1) = \inf\{t\ge 0 : \mathcal{J}_2(W_1;t)\ge 1\} \text{ if } \boldsymbol{\theta} = (-2,-1), \end{cases} \tag{16}$$

where $\inf\{\emptyset\} = \infty,\ \Lambda_2$ *is defined in (4) for $p=2$,*

$$\psi(\boldsymbol{\theta},h) = \begin{cases} (1+b)\sqrt{h/2} \text{ if } \boldsymbol{\theta}\in\Gamma_1, \\ (1-a)\sqrt{h/2} \text{ if } \boldsymbol{\theta}\in\Gamma_2, \\ \sqrt{2h}\sin\varphi \text{ if } \boldsymbol{\theta} = (2\cos\varphi,-1)'\in\Gamma_3, \\ \sqrt{2h} \text{ if } \boldsymbol{\theta} = (0,1), \\ (h/2)^{1/4} \text{ if } \boldsymbol{\theta}\in\{(-2,-1),(2,-1)\}\,, \end{cases} \tag{17}$$

$W(t), W_1(t)$ are independent standard Brownian motions.

The proofs of the theorems 2-5 are given in our paper (2008_a).

3 Asymptotic normality of the sequential LSE for AR(p).

The uniform asymptotic normality of sequential least squares estimators for the parameters in a stable AR(p) has been studied in the author paper (2004). In this section we consider the unstable model (1) with $p \geq 3$. We assume that the variance σ^2 is known and that the parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ in (1) satisfies the following conditions.

Condition 1. Parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ is such that all roots $z_i = z_i(\boldsymbol{\theta})$ of the characteristic polynomial (3) lie inside or on the unite circle.

Condition 2. All the roots $z_i = z_i(\boldsymbol{\theta})$ of $\mathcal{P}(z)$, which are equal to one in modulus, are simple.

Condition 3. The system of linear equations with respect to $Y_1, \ldots, Y_{p-1}$

$$\begin{cases} Y_1 - \sum_{l=2}^{p} \theta_l Y_{l-1} = \theta_1 \\ -\sum_{k=1}^{j-1} \theta_{j-k} Y_k + Y_j - \sum_{k=1}^{p-j} \theta_{k+j} Y_k = \theta_j, \\ 2 \leq j \leq p-1, \end{cases} \tag{18}$$

has a unique solution $(Y_1, \ldots, Y_{p-1}), Y_i = \kappa_i(\boldsymbol{\theta}), 1 \leq i \leq p-1$, and the matrix

$$L = L(\boldsymbol{\theta}) = \begin{pmatrix} 1 & \kappa_1(\boldsymbol{\theta}) & \kappa_2(\boldsymbol{\theta}) & \ldots & \kappa_{p-1}(\boldsymbol{\theta}) \\ \kappa_1(\boldsymbol{\theta}) & 1 & \kappa_1(\boldsymbol{\theta}) & \ldots & \kappa_{p-2}(\boldsymbol{\theta}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa_{p-1}(\boldsymbol{\theta}) & \kappa_{p-2}(\boldsymbol{\theta}) & \ldots & \kappa_1(\boldsymbol{\theta}) & 1 \end{pmatrix} \tag{19}$$

is positive definite.

Let $\tilde{\Lambda}_p$ denotes all $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ in (1) which satisfy all Conditions 1-3.

Theorem 6. *Suppose that in the AR(p) model (1), the parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ satisfies Conditions 1-3. Let $\boldsymbol{\theta}(\tau(h)), \tau(h)$ be defined by (7),(8). Then for any compact set $K \subset \tilde{\Lambda}_p$*

$$\lim_{h \to \infty} \sup_{\boldsymbol{\theta} \in K} \sup_{\boldsymbol{t} \in R^p} \left| P_{\boldsymbol{\theta}} \left(M_{\tau(h)}^{1/2} (\boldsymbol{\theta}(\tau(h)) - \boldsymbol{\theta}) \leq \boldsymbol{t} \right) - \Phi_p(\boldsymbol{t}/\sigma) \right| = 0\,, \tag{20}$$

where $\Phi_p(\boldsymbol{t}) = \Phi(t_1) \cdots \Phi(t_p), \Phi$ is the standard normal distribution function, $\boldsymbol{t} = (t_1, \ldots, t_p)'$.

The proof of this theorem is given in our paper (2008_b). Conditions 1-3, imposed on the parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ in (1), provide the convergence of the ratio M_n/S_n to the matrix L given in (19), where $S_n = \sum_{k=1}^{n} x_{k-1}^2$. This property can be viewed as an extension of (5) outside the stability region (4).

Lemma 1. *Let parameters $\theta_1, \ldots, \theta_p$ in the equation (1) satisfy Conditions 1-3, the $p \times p$ matrix M_n be given by (2) and L be defined in (19). Then, for any compact set $K \subset \tilde{\Lambda}_p$ and each $\delta > 0$,*

$$\lim_{m \to \infty} \sup_{\boldsymbol{\theta} \in K} P_{\boldsymbol{\theta}} \left(\| \frac{M_n}{S_n} - L \| \geq \delta \text{ for some } n \geq m \right) = 0.$$

The extension of the property of asymptotic normality of the sequential estimate to the part of the boundary $\partial \Lambda_p$ is achieved by making use of a above new property of observed Fisher information matrix M_n, which holds in a broader subset of $[\Lambda_p]$ as compared with (4).

In the conclusion it should be noted that the sequential LSE possesses two advantages over the ordinary LSE:

–its limit distribution is standard normal independent of unknown parameters;

– the normalizing factor, $M_{\tau(h)}^{1/2}$, in the limit asymptotic normality is the same for each value of the parameters.

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