One shot schemes in decentralized systems with continuous and discrete time observations

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Abstract. This work considers the problem of quickest detection with N distributed sensors that receive sequential observations either in continuous or in discrete time from the environment. These sensors employ cumulative sum (CUSUM) strategies and communicate to a central fusion center by one shot schemes. One shot schemes are schemes in which the sensors communicate with the fusion center only once, via which they signal a detection. The communication is clearly asynchronous and the case is considered in which the fusion center employs a minimal strategy, which means that it declares an alarm when the first communication takes place. It is assumed that the observations received at the sensors are independent and that the time points at which the appearance of a signal can take place are different. Both the cases of the same and different signal distributions across sensors are considered. It is shown that there is no loss of performance of one shot schemes as compared to the centralized case in an extended Lorden min-max sense, since the minimum of N CUSUMs is asymptotically optimal as the mean-time to the first false alarm increases without bound. The asymptotic optimality of the minimum of N CUSUMs is stronger in the case of different signal distributions for an appropriate choice of threshold parameters.
Keywords. One shot schemes, CUSUM, quickest detection, Optimal sensor threshold selection.

1 Description

In this work we examine the problem of quickest detection in an N-sensor network in which the information available is distributed and decentralized, a problem introduced in Veeravalli (2001) and studied by numerous authors, for example Mei (2006). We consider the situation in which the onset of a signal can occur at different times in the N sensors; that is the change points can be different for each of the N sensors. The objective is to detect the minimum of the change points. We consider both continuous Brownian motion and discrete i.i.d observations model in each sensor and assume that each sensor runs a CUSUM test. Each sensor then communicates with the central fusion center through a one shot scheme. We assume that the N sensors receive independent observations, which constitutes an assumption consistent with the fact that the N change points can be different. What we derive is asymptotic optimality of the minimum of the N CUSUMs, otherwise known as the N-CUSUM stopping rule, with respect to an appropriately selected optimality criterion.

This set-up has numerous applications especially in systems that can be captured by linear dynamic state-space models, as is typically the case in models of structural integrity (see for example Basseville et al., (2000)). Structural damage is characterized by a change in the modal parameter vector related to the eigenvalues of the state transition matrix. A key characteristic of this problem is that changes in each element of the modal parameter vector behave in a reasonably decoupled manner as seen in Basseville et al. (2007). Thus the minimum of the change points corresponding to each element in the modal parameter vector corresponds to the first time a structural damage is detected.

The paper is structured in the following way: main results are presented in section 2. In section 3, we discuss the implication of the main results to decentralized sequential detection. Finally, we conclude with some closing remarks in section 4.

2 Main results

We sequentially observe the independent processes $\{\xi_t^{(i)}; t \geq 0\}$ in a continuous model or a discrete model using sensors S_i for all $i = 1, \ldots, N$. Our continuous model is the Brownian motion model, which is a good approximation for data sampled at a high rate. In particular, we assume that each sensor S_i receives sequential observations $\{\xi_t^{(i)}; t \geq 0\}$ with $d\xi_t^{(i)} = \mu_i 1_{\{t \geq \tau_i\}} dt + dW_t^{(i)}$, where $\mu_i > 0$ and $\{W_t^{(i)}\}$ are independent standard Brownian motions. In our discrete-time observation model, we assume that before the change the $\{\xi_j^{(i)}; j = 1, 2, \ldots\}$ are i.i.d with an in-control distribution $g_0(x)$, and after the change they are i.i.d. with an out-of-control distribution $g_1^{(i)}(x)$. The assumptions satisfied are a finite Kullback-Leibler divergence and non-arithmetic log-likelihood ratios (see Tartakovsky, (2005)).

An appropriate measurable space is $\Omega = C[0,\infty) \times C[0,\infty) \times \ldots \times C[0,\infty)$ in the Brownian motion model and $\Omega = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \ldots \times \mathbb{R}^\infty$ in the discrete observation model, with $\mathcal{F} = \cup_{t \geq 0} \mathcal{F}_t$, where $\{\mathcal{F}_t\}$ is the filtration of the observation with $\mathcal{F}_t = \sigma\{(\xi_s^{(1)}, \ldots, \xi_s^{(N)}); s \leq t\}$. On the space we have the following family of probability measures $\{P_{\tau_1,\ldots,\tau_N}\}$, where $P_{\tau_1,\ldots,\tau_N}$ corresponds to the measure generated on Ω by the process $(\xi_t^{(1)}, \ldots, \xi_t^{(N)})$ when the change in the N-tuple process occurs at time point $\tau_i, i = 1, \ldots, N$. We will also consider the projection of the $P_{\tau_1,\ldots,\tau_N}$ on the i-th component of Ω, with special attention to $P_0^{(i)}$ and P_∞, for all $i = 1, \ldots, N$.

Our objective is to find a stopping rule T that balance the trade-off between a small detection delay subject to a lower bound on the mean-time to the first false alarm and will ultimately detect $\min\{\tau_1, \ldots, \tau_N\}$.

As a performance measure we consider the following generalization of Lorden's performance index

$$J^{(N)}(T) = \sup_{\substack{\tau_1,\ldots,\tau_N \\ \min_i \tau_i < \infty}} \operatorname{essup} E_{\tau_1,\ldots,\tau_N} \left\{ (T - \tau_1 \wedge \ldots \wedge \tau_N)^+ | \mathcal{F}_{\tau_1 \wedge \ldots \wedge \tau_N} \right\}, \tag{1}$$

The performance index presented in (1) results in the corresponding stochastic optimization problem of the form

$$\begin{array}{c} \inf_T J^{(N)}(T) \\ \text{subject to } E_{\infty,\ldots,\infty}\{T\} \geq \gamma. \end{array} \tag{2}$$

The first observation, whose proof is rigorously derived, is that the optimal stopping rule to problem (2) must be an equalizer rule, i.e., it must have the same detection delay regardless of the location of the first change point (see Hadjiliadis et al., (2008)).

In the case that $N = 1$, it is shown in Moustakides (1986) in discrete model, and in Beibel (1996) and Shiryaev (1996) in continuous model that the optimal stopping rule to (2) is the CUSUM stopping rule. The CUSUM stopping rule is defined as

$$T_\nu = \inf \left\{ t \geq 0; \sup_{0 \leq \tau \leq t} \log \left. \frac{dP_\tau}{dP_\infty} \right|_{\mathcal{F}_t} \geq \nu \right\}, \tag{3}$$

where $\nu > 0$ is chosen so that $E_\infty\{T_\nu\} = \gamma$. The optimality of the CUSUM stopping rule in the case $N = 1$ suggests $T_\hbar = T_{h_1}^1 \wedge T_{h_2}^2 \wedge \ldots \wedge T_{h_N}^N$ with $E_{\infty,\ldots,\infty}\{T_\hbar\}$ as a solution to (2).

Although the optimal threshold selection h_i in each sensor CUSUM is obviously the same in the case that μ_i and $g_1^{(i)}(x)$ are the same across i, decoupling the optimal threshold selection based on the equalizer rule property is far from trivial. Our first two results concern the optimal threshold selection in the non-trivial cases of different μ_i and $g_1^{(i)}(x)$'s, which are summarized in the following lemma.

Lemma 1. *Choose thresholds* $\hbar = (h_1, h_2, \ldots, h_N)$ *so that*

$$\textit{Brownian Motion: } \frac{1}{\mu_i^2}(h_i - 1) = \textit{constant}, \quad \textit{Discrete Model: } \frac{1}{I_{g_0}^{(i)}}(h_i + \beta_i + \kappa_i) = \textit{constant}, \tag{4}$$

where κ_i is related to the overshoot over the threshold which occurs in discrete time and β_i is the expectation under the measure corresponding to the out-of-control distribution of the asymptotic minimum,

$$\beta_i = E_0^{(i)} \min_{n\geq 1} \left\{ \sum_{k=1}^{n} \log \frac{g_0^{(i)}(\xi_k^{(i)})}{g_\infty(\xi_k^{(i)})} \right\}. \tag{5}$$

The constant $I_{g_0}^{(i)}$ is the Kullback-Leibler divergence, namely,

$$I_{g_0}^{(i)} = E_0^{(i)} \left\{ \log \frac{g_0^{(i)}(\xi_1^{(i)})}{g_\infty(\xi_1^{(i)})} \right\} > 0. \tag{6}$$

Then the N-CUSUM stopping rule is an equalizer rule asymptotically (hence is the best N-CUSUM stopping rule), and as $h_1 \to \infty$,

$$\textit{Brownian:}\ \ J^{(N)}(T_\hbar) = E_{0,\infty,\ldots,\infty}\{T_\hbar\} = \ldots = E_{\infty,\infty,\ldots,0}\{T_\hbar\} = \frac{1}{\mu_1^2}(h_1 - 1) + o(1), \tag{7}$$

$$\textit{Discrete:}\ \ J^{(N)}(T_\hbar) = E_{0,\infty,\ldots,\infty}\{T_\hbar\} = \ldots = E_{\infty,\infty,\ldots,0}\{T_\hbar\} = \frac{1}{I_{g_0}^{(1)}}(h_1 + \beta_1 + \kappa_1) + o(1). \tag{8}$$

Proof. Without loss of generality, we will only prove the lemma in the case $N = 2$. Let us denote by $T_{h_i}^i$, $i = 1, 2$ the CUSUM stopping rules with thresholds h_1, h_2 and by $T_\hbar = T_{h_1}^1 \wedge T_{h_2}^2$ the 2-CUSUM stopping rule. Then it suffices to show that under (4),

$$E_{0,\infty}\{T_\hbar\} = E_0^{(1)}\{T_{h_1}^{(1)}\} + o(1) = E_{\infty,0}\{T_\hbar\} + o(1) = E_0^{(1)}\{T_{h_2}^{(2)}\} + o(1), \tag{9}$$

as $h_1, h_2 \to \infty$ (see Hadjiliadis et al., (2008)). Now we observe that[1]

$$\begin{aligned} E_{0,\infty}\{T_\hbar\} &= e^{h_2} E_{0,\infty}\{e^{-h_2}T_{h_1}^1 \wedge e^{-h_2}T_{h_2}^2\} = e^{h_2}\int_0^\infty P_0^{(1)}(e^{-h_2}T_{h_1}^1 \geq t) P_\infty(e^{-h_2}T_{h_2}^2 \geq t)dt \\ &= e^{h_2}\int_0^\infty P_0^{(1)}(e^{-h_2}T_{h_1}^1 \geq t)dt - e^{h_2}\int_0^\infty P_0^{(1)}(e^{-h_2}T_{h_1}^1 \geq t)[1 - P_\infty(e^{-h_2}T_{h_2}^2 \geq t)]dt \\ &= \int_0^\infty P_0^{(1)}(T_{h_1}^1 \geq u)du - e^{h_2}\int_0^\infty P_0^{(1)}(e^{-h_2}T_{h_1}^1 \geq t)[1 - P_\infty(e^{-h_2}T_{h_2}^2 \geq t)]dt \\ &= E_0^{(1)}\{T_{h_1}^1\} - I(h_1, h_2), \end{aligned}$$

so we only need to show $I(h_1, h_2) \to 0$ as $h_1, h_2 \to \infty$ under (4). In both the Brownian model and the discrete model, it is known that for large h_2, (see Taylor, (1975) and Tartakovsky, (2005))

$$P_\infty(e^{-h_2}T_{h_2}^2 \geq t) = [1 + o(1)]e^{-t}, \tag{10}$$

so we can estimate $I(h_1, h_2)$ by

$$e^{h_2}\int_0^\infty P_0^{(1)}(e^{-h_2}T_{h_1}^1 \geq t)\left[1 - [1 + o(1)]e^{-t}\right]dt = [1 + o(1)]\int_0^\infty P_0^{(1)}(T_{h_1}^1 \geq u)(1 - e^{-ue^{-h_2}})du.$$

By using the fact that $1 - e^{-x} \leq x$ we further have

$$0 \leq I(h_1, h_2) \leq [1 + o(1)]\int_0^\infty P_0^{(1)}(T_{h_1}^1 \geq u)ue^{-h_2}du = \frac{1 + o(1)}{2}e^{-h_2}E_0^{(1)}\{(T_{h_1}^1)^2\}.$$

However, it can be easily shown that $E_0^{(1)}\{(T_{h_1}^1)^2\} = O((h_1)^2)$ (see Taylor, (1975) and Tartakovsky, (2005)), thus $I(h_1, h_2) \to 0$ as $h_1, h_2 \to \infty$. This shows the first equality in (9), and a similar argument shows the last equality in (9). The second equality follows immediately from (4) and the asymptotic expansion of the expectation of the CUSUM stopping rule. □

[1] The integral representation is used for convenience. However, it should be realized that every integral is actually a summation.

In our related paper, Hadjiliadis et al. (2008), it is seen that with the above threshold selection, if k-out-of-N of the constants μ_i are smallest and equal to each other, without loss of generality, let $\mu_1 = \min_i \mu_i$, then the difference in detection delay of the unknown optimal stopping rule and N-CUSUM is bounded above by the constant $\frac{2}{\mu_1^2} \log k$ as $\gamma \to \infty$. One of the main implications of this fact is that when all of the μ_i's are different this difference is bounded above by 0 as $\gamma \to \infty$.

Similarly, in the discrete-time model, if we let $I_{g_0}^{(1)} < \min_{i>1}\{I_{g_0}^{(i)}\}$, the difference in detection delay of the N-CUSUM stopping rule and the unknown optimal stopping scheme tends to 0 as $\gamma \to \infty$. Now, consider the more general case in which

$$I_{g_0}^{(1)} = I_{g_0}^{(2)} = \ldots = I_{g_0}^{(k)} < \min_{i>k}\{I_{g_0}^{(i)}\}. \tag{11}$$

Without loss of generality, we also assume that

$$(R_1)^2 e^{\beta_1+\kappa_1} = \max_{1\le i\le k}\{(R_i)^2 e^{\beta_i+\kappa_i}\}, \tag{12}$$

where R_i are also constants related to the overshoot of the threshold under the out-of-control distribution. In this case we have

Theorem 1. *The difference in detection delay of the N-CUSUM stopping rule and the unknown optimal stopping rule is bounded above, as $\gamma \to \infty$, by the constant*[2]

$$\frac{1}{I_{g_0}^{(1)}} \log\left[\sum_{i=1}^{k}\left(\frac{R_i}{R_1}\right)^2 r_i\right] \le \frac{1}{I_{g_0}^{(1)}} \log k, \ \textit{with } r_i = e^{(\beta_i-\beta_1)+(\kappa_i-\kappa_1)}. \tag{13}$$

To prove Theorem 1, we need the asymptotic expansion of the mean-time to the first false alarm for the N-CUSUM stopping rule as thresholds $(h_1, \ldots, h_N)$ are chosen so that (4) holds. And this is presented in the next lemma.

Lemma 2. *Under condition (4), we have*

$$E_{\infty,\ldots,\infty}\{T_{\hbar}\} = \left(\sum_{i=1}^{N} I_{g_0}^{(i)}(R_i)^2 e^{-h_i}\right)^{-1}[1+o(1)] = \frac{e_1^h}{I_{g_0}^{(1)}\sum_i^k (R_i)^2 r_i}[1+o(1)], \tag{14}$$

as $h_i \to \infty$.

Proof. By using Lemma 1 of Tartakovsky (2005), the first equality follows. To see the last equality, note that under (4),

$$\begin{aligned}
\left(\sum_{i=1}^{N} I_{g_0}^{(i)}(R_i)^2 e^{-h_i}\right)^{-1} &= \left(\sum_{i=1}^{k} I_{g_0}^{(i)}(R_i)^2 e^{-h_i} + \sum_{i=k+1}^{N} I_{g_0}^{(i)}(R_i)^2 e^{-h_i}\right)^{-1} \\
= \left(\sum_{i=1}^{k} I_{g_0}^{(i)}(R_i)^2 e^{-h_i}\right)^{-1} &[1+o(1)] = \frac{e^{h_1}}{I_{g_0}^{(1)}\sum_{i=1}^k (R_i)^2 e^{(\beta_i-\beta_1)+(\kappa_i-\kappa_1)}}[1+o(1)] \\
= \frac{e^{h_1}}{I_{g_0}^{(1)}\sum_{i=1}^k (R_i)^2 r_i}&[1+o(1)],
\end{aligned}$$

as $h_i \to \infty$. □

Let us proceed to the proof of Theorem 1.

[2] It shows that we have a sharper upper bound in the discrete model than that in the Brownian model.

Proof (of Theorem 1). First we observe that, for our particularly chosen N-CUSUM stopping rule, we have

$$J^{(N)}(T_{\hbar}) > J^{(N)}(T^{*}) > \max_{1\le i\le N}\{E_0^{(i)}\{T_{\nu_i}^{i}\}\}, \tag{15}$$

where T^{*} is the unknown optimal stopping rule to problem (2), and $\{\nu_i\}_{i=1}^{N}$ are chosen so that $E_{\infty}\{T_{\nu_i}^{i}\} = \gamma$. By our assumptions (11) and (12), the asymptotic lower bound in (15) is (see Tartakovsky, (2005))

$$E_0^{(1)}\{T_{\nu_1}^{1}\} = \frac{1}{I_{g_0}^{(1)}}\left[\log\gamma + \log\left(I_{g_0}^{(1)}(R_1)^2\right) + \beta_1 + \kappa_1\right] + o(1). \tag{16}$$

However, Lemma 1 and Lemma 2 imply that, as $\gamma\to\infty$,

$$J^{(N)}(T_{\hbar}) = \frac{1}{I_{g_0}^{(1)}}\left[\log\gamma + \log\left(I_{g_0}^{(1)}\sum_{i=1}^{k}(R_i)^2 r_i\right) + \beta_1 + \kappa_1\right] + o(1). \tag{17}$$

Thus,

$$J^{(N)}(T_{\hbar}) - J^{(N)}(T^{*}) \le J^{(N)}(T_{\hbar}) - E_0^{(1)}\{T_{\nu_1}^{1}\} = \frac{1}{I_{g_0}^{(1)}}\log\left[\sum_{i=1}^{k}\left(\frac{R_i}{R_1}\right)^2 r_i\right] + o(1), \tag{18}$$

as $\gamma\to\infty$. $\square$

For the equivalent of Theorem 1 and Lemma 2 in Brownian motion model, please refer to see Hadjiliadis et al. (2008).

3 Decentralized detection

Let us now suppose that each of the observation processes $\{\xi_t^{(i)}\}$ become sequentially available at its corresponding sensor S_i, which then employs an asynchronous communication scheme to the central fusion center. In particular, sensor S_i communicates to the central fusion center only when it wants to signal an alarm, which is elicited according to a CUSUM rule $T_{h_i}^{i}$. Once again the observations received at the N sensors are independent and can change dynamics at distinct unknown points τ_i. An example of such a case is described in Basseville et al. (2007) where the motivation suggested arises in the health-monitoring of mechanical, civil and aeronautic structures. In this treatment the vibration-based and health monitoring problems translate into the identification and monitoring of the eigenstructure of a state transition matrix of a linear dynamical state-space system excited by noise (see Basseville et al., (2000) and (2007)). This is achieved in practice by detecting a change in the canonical modal parameter vector associated with the eigenstructure. In Basseville et al. (2007) it is characteristically pointed out that the individual subspace-based tests, monitoring each parameter-vector component, appear to behave in a reasonably decoupled manner and to perform a correct isolation of the components of the vector parameter that has changed. Thus each in this set-up the distinct change points τ_i correspond to the change points of the value of each parameter-vector component. The decoupled manner in which each parameter-vector component behaves corresponds to the fact that there is absence of across-sensor correlations. The fusion center, whose objective is to detect the first time when there is a change in at least one of the parameter-vector components, devises a minimal strategy; that is, it declares that a change has occurred at the first instance when one of the sensors communicates an alarm. The implication of Theorem 1 is that in fact this strategy is the best, at least asymptotically, that the fusion center can devise and that there is no loss in performance, between the case in which the fusion center receives the raw data $(\xi_t^{(1)},\ldots\xi_t^{(N)})$ directly and the case in which the communication that takes place only when any sensor signals an alarm. To see this, consider the general case in which the first k out of N sensors receive the same signal strength after the onset of a signal or equivalently in discrete time the case in which k out of the N out-of-control distributions are the same . Then the rule suggested by Theorem 1 is $T_{\hbar} = T_h^1 \wedge \ldots \wedge T_h^k \wedge \ldots \wedge T_{h_{k+1}}^{k+1} \wedge \ldots \wedge T_{h_N}^N$ with $\hbar = (h, \ldots, h, h_{k+1}, \ldots, h_N)$ so that, at least asymptotically, the N-CUSUM stopping rule is an equalizer rule. Thus the detection delay of $T_{\hbar}$ is the same, at least asymptotically, regardless of which of the sensors S_i draws the alarm of detection first. The mean-time to the first false alarm for the fusion center that uses the rule $T_{\hbar}$ is thus $E_{\infty,\ldots,\infty}\{T_{\hbar}\}$. But Theorem 1 asserts that this rule, namely $T_{\hbar}$, is asymptotically optimal as the mean-time to the first false alarm tends to ∞ in the centralized case for any finite N. In other words, the CUSUM stopping rule $T_{\hbar}$ is a sufficient statistic (at least asymptotically) of the minimum N possibly distinct change points. That is, the stopping rule $T_{\hbar}$ is an asymptotically optimal solution to the problems of quickest detection presented in (2).

4 Conlusion

The main contribution of this paper is that it shows that one can distribute most of the work of change detection in sensor network to the sensors without any loss of performance at least asymptotically both in the case of continuous-time models and in the case of discrete-time models. The applications of this set-up are numerous and rely on the detection of the individual components in a vector parameter corresponding to the eigenstructure of linear dynamical state-space models. Such models have been extensively used to describe for monitoring the health of mechanical, civil and aeronautical structures (see Basseville et al., (2000) and (2007)). The assumption of across-sensor independence is realistic at least in the particular examples which are described in detail in Basseville et al. (2007). Moreover, the set-up treated in this paper is also relevant to the case in which the change points propagate in a sensor array in Raghavan and Veeravalli (2008). This is because even in this configuration the propagation of the change points depends on the unknown identity of the first sensor affected. In our paper we give explicit formulas for the optimal sensor threshold selection which becomes particularly relevant in the general case in which the observation out-of-control distributions or the signal strengths are different across sensors.

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