Optimal Sequential Procedures with Bayes Decision Rules

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Abstract. In this article, a general problem of sequential statistical inference for general discrete-time stochastic processes is considered. Let $X_1, X_2, \dots$ be a discrete-time stochastic process, whose distribution depends on an unknown parameter θ, $\theta \in \Theta$. We consider a problem of optimal sequential decision-making in the following framework. Let $w_n(\theta, d; x_1, \dots, x_n) \geq 0$, $\theta \in \Theta$, $d \in \mathscr{D}$, be a loss function representing losses from making a decision d at stage n of a statistical experiment, when the true parameter value is θ, and the data observed up to this stage are $x_1, \dots, x_n$. Let $K_\theta^n(x_1, \dots, x_n)$ be the cost of the observations when θ is the true value of the parameter. The decision is supposed to be taken through a sequential decision-making procedure (τ, δ), where τ is a stopping time with respect to the sequence of σ-algebras $\mathscr{F}_n = \sigma(X_1, X_2, \dots, X_n)$, $n = 1, 2, \dots$, and δ is an $\mathscr{F}_\tau$-measurable decision function with values in $\mathscr{D}$. For any sequential decision procedure (τ, δ) let us define the average loss due to incorrect decision

$$W(\theta; \tau, \delta) = E_\theta w_\tau(\theta, \delta; X_1, \dots X_\tau),$$

and the average cost of observations as

$$C(\theta; \tau) = E_\theta K_\theta^\tau(X_1, \dots, X_\tau).$$

Let, finally, the "risk function" be defined as

$$R(\tau, \delta) = \int_\Theta W(\theta; \tau, \delta) d\pi_1(\theta) + \int_\Theta C(\theta; \tau) d\pi_2(\theta),$$

where π_1 and π_2 are some probability measures on Θ. The main goal of this article is to give conditions of existence of sequential decision procedures which minimize $R(\tau, \delta)$ (optimal decision procedures), and characterize their structure. In particular, when $\pi_1 = \pi_2 = \pi$ is an *a priori* distribution of the parameter, we give a characterization of optimal (Bayesisan) sequential decision procedures minimizing $R(\tau, \delta)$ among all sequential decision procedures (τ, δ).

Keywords. Bayes decision, dependent observations, discrete-time stochastic process, optimal decision rule, optimal stopping rule, randomized stopping time, sequential analysis, statistical decision problem.

1 Introduction

Let $X_1, X_2, \dots, X_n, \dots$ be a discrete-time stochastic process, whose distribution depends on an unknown "parameter" θ, $\theta \in \Theta$. In this article, we consider a general problem of sequential statistical decision making based on the observations of this process.

Let us define a *sequential statistical procedure* as a pair (ψ, δ), being ψ a (randomized) *stopping rule*, $\psi = (\psi_1, \psi_2, \dots, \psi_n, \dots)$, and δ a *decision function*, $\delta = (\delta_1, \delta_2, \dots, \delta_n, \dots)$, supposing that $\psi_n = \psi_n(x_1, x_2, \dots, x_n)$ and $\delta_n = \delta_n(x_1, x_2, \dots, x_n)$ are measurable functions, and $\psi_n(x_1, \dots, x_n) \in [0, 1]$, $\delta_n(x_1, \dots, x_n) \in \mathscr{D}$ for any observations vector $(x_1, \dots, x_n)$, for any $n = 1, 2, \dots$ (see, for example, Wald, 1950, Ferguson, 1967, Ghosh et al., 1997.

For any stage number $n \geq 1$, $\psi_n(x_1, \dots, x_n)$ is interpreted as the conditional probability to stop and proceed to decision making, given that we did not stop before and that the observations up to this stage were $(x_1, \dots, x_n)$, and $\delta_n(x_1, \dots, x_n)$ as the decision to take when stopping occurs, $n = 1, 2, \dots$.

The stopping rule ψ generates a random variable τ_ψ (*stopping time*) whose distribution is given by

$$P_\theta(\tau_\psi = n) = E_\theta(1 - \psi_1)(1 - \psi_2) \dots (1 - \psi_{n-1})\psi_n, \quad n = 1, 2, \dots \tag{1}$$

(here, and in what follows, we interchangeably use ψ_n both for $\psi_n(x_1, x_1, \dots, x_n)$ and for $\psi_n(X_1, X_1, \dots, X_n)$: it ψ_n is under the expectation or probability sign, then it is $\psi_n(X_1, \dots, X_n)$, otherwise it is $\psi_n(x_1, \dots, x_n)$).

Let $w_n(\theta, d; x_1, \ldots, x_n)$ be a non-negative loss function, $n = 1, 2, \ldots$ (we suppose that w_n is a measurable function of all its arguments for any $n \geq 1$). Let π_1 be any probability measure. We define the average loss of the sequential statistical procedure (ψ, δ) due to wrong decision as

$$W(\psi, \delta) = \sum_{n=1}^{\infty} \int \left[E_\theta (1 - \psi_1) \ldots (1 - \psi_{n-1}) \psi_n w_n(\theta, \delta_n; X_1, \ldots, X_n) \right] d\pi_1(\theta). \tag{2}$$

Let also $K_\theta^n = K_\theta^n(x_1, \ldots, x_n)$ be a non-negative (and measurable with respect to $(\theta, x_1, \ldots, x_n)$) cost function, $n \geq 1$, such that $K_\theta^n(x_1, \ldots, x_n) \leq K_\theta^{n+1}(x_1, \ldots, x_n, x_{n+1})$ for any observation sequence $x_1, x_2, \ldots, x_{n+1}$,$n \geq 1$, $\theta \in \Theta$.

Let us define the *average cost* of the sequential decision procedure (τ, δ) as

$$C(\theta; \psi) = E_\theta K_\theta^{\tau_\psi}(X_1, \ldots, X_{\tau_\psi}) \tag{3}$$

(we suppose that $K(\theta; \psi) = \infty$ if $\sum_{n=1}^{\infty} P_\theta(\tau_\psi = n) < 1$, see (1)).

Let us also define a “weighted” value of the average cost

$$C(\psi) = \int C(\theta; \psi) d\pi_2(\theta), \tag{4}$$

where π_2 is some probability measure giving “weights” to particular values of θ.

Our main goal is minimizing the “weighted risk”

$$R(\psi, \delta) = C(\psi) + W(\psi, \delta), \tag{5}$$

supposing that π_1 in (2) and π_2 in (4) are, generally speaking, two *different* probability measures. If $\pi_1 = \pi_2 = \pi$, $R(\psi, \delta)$ is called *Bayesian risk* of (ψ, δ) corresponding to the *a priori* distribution π (see, for example, Wald and Wolfowitz, 1948, Wald, 1950, Ferguson, 1967, Schmitz, 1993, Ghosh et al. (1997), among many others).

To guarantee that $\inf R(\psi, \delta)$ is finite we suppose that $\inf_\delta R(\psi^1, \delta) < \infty$ with $\psi^1 = (1, \ldots)$.

We use essentially the same method as in Novikov (2008), where the case of $K_\theta^n \equiv n$ and $w_n(\theta, d; x_1, \ldots, x_n) \equiv w(\theta, d)$ for any $\theta \in \Theta$, $d \in \mathscr{D}$, and for any $(x_1, \ldots, x_n)$, $n \geq 1$, was considered. In turn, the method of Novikov (2008) is an extension of the results of Novikov (2009).

2 Main results

Throughout the paper we suppose that for any $n = 1, 2, \ldots$, the vector $(X_1, X_2, \ldots, X_n)$ has a probability “density” function

$$f_\theta^n = f_\theta^n(x_1, x_2, \ldots, x_n) \tag{6}$$

(Radon-Nikodym derivative of its distribution) with respect to a product-measure

$$\mu^n = \underbrace{\mu \otimes \mu \otimes \cdots \otimes \mu}_{n \quad \text{times}},$$

with some σ-finite measure μ on the respective space. As usual in the Bayesian context, we suppose that $f_\theta^n(x_1, x_2, \ldots, x_n)$ is measurable with respect to $(\theta, x_1, \ldots, x_n)$, for any $n = 1, 2, \ldots$.

Let us suppose that for any $n \geq 1$ there exists a measurable $\delta_n^B = \delta_n^B(x_1, \ldots, x_n)$ such that for any $d \in \mathscr{D}$

$$\int w_n(\theta, d; x_1, \ldots, x_n) f_\theta^n(x_1, \ldots, x_n) d\pi_1(\theta) \geq \int w_n(\theta, \delta_n^B; x_1, \ldots, x_n) f_\theta^n(x_1, \ldots, x_n) d\pi_1(\theta) \tag{7}$$

for all data sequences $(x_1, \ldots, x_n)$. Let $\delta^B = (\delta_1^B, \delta_2^B, \ldots, \delta_n^B, \ldots)$. It is easy to see that in this case for any decision function $\delta_n = \delta_n(x_1, \ldots, x_n)$

$$\int_\Theta E_\theta w_n(\theta, \delta; X_1, \ldots, X_n) d\pi_1(\theta) \geq \int_\Theta E_\theta w_n(\theta, \delta_n^B; X_1, \ldots, X_n) d\pi_1(\theta),$$

i.e. δ_n^B is a *Bayesian decision function* (corresponding to the “a priori” distribution π_1) based on n observations.

Let us denote $l_n = l_n(x_1, \dots, x_n)$ the right-hand side of (7). From this time on, we suppose that $\int l_n d\mu_n < \infty$ for any $n = 1, 2, \dots$.

In the same way as in Novikov (2008) we easily get

Theorem 1. *For any sequential decision procedure* (ψ, δ)

$$W(\psi, \delta) \geq W(\psi, \delta^B) = \sum_{n=1}^{\infty} \int (1-\psi_1)\dots(1-\psi_{n-1})\psi_n l_n d\mu^n. \tag{8}$$

It follows from Theorem 1 that $\inf_\delta W(\psi, \delta) = W(\psi, \delta^B)$, and the aim of what follows is to minimize

$$L(\psi) = C(\psi) + W(\psi, \delta^B)$$

over all stopping rules ψ (see (5)) .

It is easy to see that, by definition of $C(\psi)$,

$$L(\psi) = \sum_{n=1}^{\infty} \int (1-\psi_1)\dots(1-\psi_{n-1})\psi_n \left(\int K_\theta^n f_\theta^n d\pi_2(\theta) + l_n \right) d\mu^n \tag{9}$$

if $\int P_\theta(\tau_\psi < \infty) d\pi_2(\theta) = 1$, and $L(\psi) = \infty$ otherwise.

Let us denote

$$k_n = k_n(x_1, \dots, x_n) = \int K_\theta^n(x_1, \dots, x_n) f_\theta^n(x_1, \dots, x_n) d\pi_2(\theta)$$

(see (9)), and let for any $\pi = \pi_1$ or $\pi = \pi_2$ $P^\pi(A) = \int P_\theta(A) d\pi(\theta)$ for any event A.

Let also

$$s_n^\psi = s_n^\psi(x_1, \dots, x_n) = (1-\psi_1(x_1))\dots(1-\psi_{n-1}(x_1, \dots, x_{n-1}))\psi_n(x_1, \dots, x_n)$$

for any $n = 1, 2, \dots$ and for any stopping rule ψ.

Thus, by (9),

$$L(\psi) = \sum_{n=1}^{\infty} \int s_n^\psi (k_n + l_n) d\mu^n$$

if $P^{\pi_2}(\tau < \infty) = 1$, and $L(\psi) = \infty$ otherwise.

First, let us solve the problem of minimization of $L(\psi)$ in the class $\mathscr{F}^N$ of truncated stopping rules, that is such that $\psi = (\psi_1, \psi_2, \dots, \psi_{N-1}, 1, \dots)$, $N = 2, 3, \dots$ (see also Novikov, 2008).

For any $\psi \in \mathscr{F}^N$ let

$$L_N(\psi) = \sum_{n=1}^{N} \int s_n^\psi (k_n + l_n) d\mu^n = \sum_{n=1}^{N-1} \int s_n^\psi (k_n + l_n) d\mu^n + \int c_N^\psi (k_N + l_N) d\mu^N, \tag{10}$$

where, for any $n \geq 1$ and for any stopping rule ψ

$$c_n^\psi = c_n^\psi(x_1, \dots, x_n) = (1-\psi_1(x_1))\dots(1-\psi_{n-1}(x_1, \dots, x_{n-1})).$$

Theorem 2. *Let* $\psi \in \mathscr{F}^N$ *be any (truncated) stopping rule,* $N \geq 2$*. Then for any* $1 \leq r \leq N-1$ *the following inequalities hold true*

$$L_N(\psi) \geq \sum_{n=1}^{r} \int s_n^\psi (k_n + l_n) d\mu^n + \int c_{r+1}^\psi \left(k_{r+1} + V_{r+1}^N\right) d\mu^{r+1} \tag{11}$$

$$\geq \sum_{n=1}^{r-1} \int s_n^\psi (k_n + l_n) d\mu^n + \int c_r^\psi \left(k_r + V_r^N\right) d\mu^r, \tag{12}$$

where $V_N^N \equiv l_N$*, and recursively for* $m = N-1, N-2, \ldots 1$

$$V_m^N = \min\{l_m, Q_m^N\}, \tag{13}$$

where

$$Q_m^N = \int \left(k_{m+1} + V_{m+1}^N\right) d\mu(x_{m+1}) - k_m \tag{14}$$

(it should be remembered that the function under the integral sign on the right-hand side of (14) is a function of $(x_1, \ldots, x_{m+1})$*, and, because of this,* $Q_m^N = Q_m^N(x_1, \ldots, x_m)$*).*

The lower bound in (12) is attained if and only if

$$I_{\{l_m < Q_m^N\}} \leq \psi_m \leq I_{\{l_m \leq Q_m^N\}} \tag{15}$$

μ^m*-almost everywhere on*

$$C_m^\psi = \{(x_1, \ldots, x_m) :\, c_m^\psi(x_1, \ldots, x_m) > 0\},$$

for any $m = r, r+1, \ldots, N-1$*.*

In particular, $(\psi_1, \psi_2, \ldots, \psi_{N-1}, 1, \ldots)$ *is an optimal truncated stopping rule in* $\mathscr{F}^N$*, if and only if (15) is satisfied* μ^m*-almost everywhere on* C_m^ψ *for any* $m = 1, \ldots, N-1$*. In addition,*

$$\inf_{\psi \in \mathscr{F}^N} L(\psi) = \int \left(k_1(x) + V_1^N(x)\right) d\mu(x). \tag{16}$$

Proof. The proof can be implemented by induction as in the proof of Theorem 3 in Novikov (2008) using instead of Lemma 2 of Novikov (2008) the following extension of it.

Lemma 1. *Let* $r \geq 1$ *be any natural number, and let* $v_{r+1} = v_{r+1}(x_1, x_2, \ldots, x_{r+1})$ *be any non-negative measurable function, such that* $\int v_{r+1} d\mu^{r+1} < \infty$*. Then*

$$\sum_{n=1}^{r} \int s_n^\psi (k_n + l_n) d\mu^n + \int c_{r+1}^\psi \left(k_{r+1} + v_{r+1}\right) d\mu^{r+1} \geq \sum_{n=1}^{r-1} \int s_n^\psi (k_n + l_n) d\mu^n + \int c_r^\psi \left(k_r + v_r\right) d\mu^r, \tag{17}$$

where $v_r = \min\{l_r, Q_r\}$*, with*

$$Q_r = Q_r(x_1, \ldots, x_r) = \int \left(k_{r+1}(x_1, \ldots, x_{r+1}) + v_{r+1}(x_1, \ldots, x_{r+1})\right) d\mu(x_{r+1}) - k_r(x_1, \ldots, x_r).$$

There is an equality in (17) if and only if $I_{\{l_r < Q_r\}} \leq \psi_r \leq I_{\{l_r \leq Q_r\}}$ μ^r*-almost everywhere on* C_r^ψ*.*

Proof of Lemma 1 can be implemented following the steps of the proof of Lemma 2 in Novikov (2008) and is omitted here.

Starting with the class of non-truncated stopping rule, let us define for any ψ

$$L_N(\psi) = \sum_{n=1}^{N-1} \int s_n^\psi \left(k_n + l_n\right) d\mu^n + \int c_N^\psi \left(k_N + l_N\right) d\mu^N.$$

The idea of construction of optimal stopping rules is to pass to the limit, as $N \to \infty$, in (11), (12), (13) and (14).

Let $\mathscr{F}$ be a class of stopping times such that for every $\psi \in \mathscr{F}$

$$P^{\psi_2}(\tau_\psi < \infty) = 1 \quad \text{and} \quad \lim_{N \to \infty} L_N(\psi) = L(\psi).$$

In a very similar manner as in Novikov (2008) it can be shown that for any $m = 1, 2, \ldots$ y any $N \geq m$ $V_m^N(x_1, \ldots, x_m) \geq V_m^{N+1}(x_1, \ldots, x_m)$ for any $(x_1, \ldots, x_m)$, so there exists

$$V_m = V_m(x_1, \ldots, x_m) = \lim_{N \to \infty} V_m^N(x_1, \ldots, x_m).$$

Thus, passing to the limit, for any $\psi \in \mathscr{F}$, in (11), (12), (13) and (14) is justified by the Lebesgue's monotone convergence theorem. In particular, let

$$Q_m = Q_m(x_1, \dots, x_m) = \lim_{N \to \infty} Q_m^N(x_1, \dots, x_m).$$

In the same way as in Novikov (2008) it can be shown that $\inf_{\psi \in \mathscr{F}} L(\psi) = \int (k_1(x) + V_1(x))\, d\mu(x)$ (cf. (16)).

Combining all these ideas, we immediately have

Theorem 3. *If there exists $\psi \in \mathscr{F}$ such that*

$$L(\psi) = \inf_{\psi' \in \mathscr{F}} L(\psi') \tag{18}$$

then

$$I_{\{l_m < Q_m\}} \leq \psi_m \leq I_{\{l_m \leq Q_m\}} \tag{19}$$

μ^m-almost everywhere on C_m^ψ, for any $m = 1, 2, \dots$.

On the other hand, if ψ satisfies (19) μ^m-almost everywhere on C_m^ψ, for any $m = 1, 2, \dots$, and $\psi \in \mathscr{F}$ then it satisfies (18) as well.

Proof. The proof can be conducted following the steps of the proof of Theorem 4 in Novikov (2008), using Lemma 1 instead of Lemma 2 of Novikov (2008).

Very much like in Novikov (2008), we can give some conditions, under which the structure of (19) is necessary and sufficient for optimality in the class of all stopping rules.

Let us call the problem of minimizing $L(\theta)$ *truncatable* if for any ψ such that $P^{\pi_2}(\tau_\psi < \infty) = 1$ it holds $L_N(\psi) \to L(\psi)$, as $N \to \infty$.

Theorem 4. *Let the problem of minimizing $L(\theta)$ be truncatable, and let for any $c > 0$*

$$\int P_\theta(K_\theta^n(X_1, \dots, X_n) < c) d\pi_2(\theta) \to 0 \quad \textit{as} \quad n \to \infty. \tag{20}$$

Then

$$L(\psi) = \inf_{\psi'} L(\psi') \tag{21}$$

if and only if

$$I_{\{l_m < Q_m\}} \leq \psi_m \leq I_{\{l_m \leq Q_m\}} \tag{22}$$

μ^m-almost everywhere on C_m^ψ, for any $m = 1, 2, \dots$.

Proof. The "if"-part can be proved analogously to the proof of Theorem 4 in Novikov (2008), using Lemma 1 instead of Lemma 2 in Novikov (2008).

To prove the "only if"-part we suppose that ψ satisfies (19) μ^m-almost everywhere on C_m^ψ, for any $m = 1, 2, \dots$. It follows from Lemma 1 that for any fixed $m = 1, 2, \dots$

$$\sum_{n=1}^{m-1} \int s_n^\psi (k_n + l_n) d\mu^n + \int c_m^\psi (k_m + V_m) d\mu^m = \int (k_1(x) + V_1(x)) d\mu(x) = I < \infty. \tag{23}$$

In particular, this implies that $\int c_m^\psi k_m d\mu^m \leq I$, or

$$\int E_\theta c_m^\psi K_\theta^m d\pi_2(\theta) \leq I, \tag{24}$$

where $c_m^\psi = c_m^\psi(X_1, \dots, X_m)$ and $K_\theta^m = K_\theta^m(X_1, \dots, X_m)$.

Let C be any positive constant. Then (24) implies

$$C \int E_\theta c_m^\psi I_{\{K_\theta^m > C\}} d\pi_2(\theta) < I, \quad m = 1, 2, \dots. \tag{25}$$

Because

$$\int E_\theta c_m^\psi d\pi_2(\theta) = \int E_\theta c_m^\psi I_{\{K_\theta^m > C\}} d\pi_2(\theta) + \int E_\theta c_m^\psi I_{\{K_\theta^m \le C\}} d\pi_2(\theta) \tag{26}$$

and the second summand by virtue of (20) tends to 0, as $m \to \infty$, we have that the difference between the first summand on the right-hand side of (26) and the left-hand side of it, goes to 0 as $m \to \infty$. Thus, from (25), we have that

$$\lim_{m\to\infty} \int E_\theta c_m^\psi d\pi_2(\theta) = \lim_{m\to\infty} \int P_\theta(\tau_\psi \ge m) d\pi_2(\theta) = \int P_\theta(\tau_\psi = \infty) d\pi_2(\theta) < I/C, \tag{27}$$

and, because of arbitrariness of C, $P^{\pi_2}(\tau = \infty) = 0$, or

$$P^{\pi_2}(\tau < \infty) = 1. \tag{28}$$

Now, from (23) we get that

$$\lim_{m\to\infty} \sum_{n=1}^{m-1} \int s_n^\psi (k_n + l_n) d\mu^n = L(\psi) \le I. \tag{29}$$

Because the problem is truncatable, it follows from (28) that $L_N(\psi) \to L(\psi)$, as $N \to \infty$. Now, passing to the limit in (16), we get $L(\psi) \ge I$. From this and (29) it follows that $L(\psi) = I = \inf_{\psi'} L(\psi')$.

Very much like in Novikov (2008) (see Corollary 1 therein), there are simple conditions which guarantee that the problem is truncatable.

Proposition 1. *The problem of minimization of $L(\psi)$ is truncatable if any of the following two conditions is fulfilled.*

(i) *There is M, $0 < M < \infty$ such that $w_n(\theta, d; x_1, \dots, x_n) \le M$ for any $\theta, d, x_1, \dots, x_n$, and for any $n \ge 1$, and from $L(\psi) < \infty$ it follows that*

$$P^{\pi_1}(\tau_\psi < \infty) = 1.$$

(ii)

$$\int l_n d\mu^n \to 0, \quad \textit{as} \quad n \to \infty.$$

Proposition 1 can be proved in the same way as Corollary 1 in Novikov (2008).

Combining Theorem 1 with Theorem 2 or Theorem 3 or Theorem 4, we have, under respective conditions, sequential decision procedures (ψ, δ^B) minimizing $R(\psi, \delta)$ in the corresponding class of sequential decision procedures, and the respective necessary conditions under which the minimum is attained.

References

Ferguson, T. (1967). *Mathematical Statistics: A Decision Theoretic Approach.* Probability and Mathematical Statistics, Vol. 1. Academic Press, New York-London.

Ghosh, M., Mukhopadhyay, N. and Sen, P. K. (1997). *Sequential Estimation*. John Wiley & Sons, New York-Chichester-Weinheim-Brisbane-Singapore-Toronto.

Novikov, A. (2008). Optimal Sequential Procedures With Bayes Decision Rules, preprint arXiv:0812.0159v1 [math.ST] (http://arxiv.org/abs/0812.0159).

Novikov, A. (2009). Optimal Sequential Tests for Two Simple Hypotheses, *Sequential Analysis* **28**, no. 2.

Schmitz, N. (1993). *Optimal Sequentially Planned Decision Procedures.* Lecture Notes in Statistics, 79. Springer-Verlag, New York.

Wald, A. (1950). *Statistical Decision Functions*. John Wiley & Sons, Inc., New York-London-Sydney.

Wald, A. and Wolfowitz, J. (1948). Optimum Character of the Sequential Probability Ratio Test. *Ann. Math. Statistics* **19**, 326–339.