FBSDEs with stopping times and optimal stopping problems

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Abstract. In this note, under certain conditions we prove an a priori estimate and a uniqueness result for a quadratic FBSDE related to a bidimensional optimal stopping problem: Find a stopping time τ^* and a value function $\Phi(x, y)$ such that

$$\Phi(x,y) = \sup_{\tau} \mathbb{E}^{x,y} \left[x_\tau - \int_0^\tau c(y_s)ds - H(y_\tau) \right],$$

where the supremum is taken over all finite stopping times, $c(.)$, $H(.)$ are positive continuous functions such that $H(.)$ is bounded, and $Q_t = (x_t, y_t)$ is a weakly coupled geometric Brownian motion. The present result gives a new explicit characterization of the above value function, as a kind of FBSDE of the quadratic type with a finite stopping time almost surely. This extends a recent result of the author.

Keywords. forward-backward stochastic differential equation; optimal stopping problems; quasilinear elliptic equations.

1. Introduction

Nonlinear BSDEs were first introduced in Pardoux and Peng (1990), under Lipschitz conditions on the coefficient and with a square integrable terminal condition. Recently, there has been some interest in a new class of quadratic BSDEs, see Lepeltier and San Martin (1998), Kobylanski (2000), Bahlali et al. (2002), Briand and Hu (2006). This class has several possible applications in mathematical finance, see El Karoui and Rouge (2000), Hu et al. (2005) and Sekine (2006) . The main motivation in the study of such type of BSDEs stems from their strong connection with quasilinear equations with quadratic growth in the gradient, see for instance Boccardo et al. (1982,1988), Donato and Giachetti (1986), Barles and Murat (1995). In this note, we prove an a priori estimate and a uniqueness result for a quadratic FBSDE with a finite stopping time almost surely, under non-Lipschitz conditions. The present result gives a new explicit characterization of a bidimensional optimal stopping problem described by a weakly coupled, geometric Brownian motion. This extends our recent result (see Makasu (2008)).

2. Optimal stopping of a bidimensional diffusion process

In this section, we consider a bidimensional optimal stopping problem described by a weakly coupled, non-degenerate diffusion process. The present problem generalizes our recent result (see Makasu (2008)). We shall consider a problem of finding the optimal time to invest in a given stock under stochastic volatility with the goal of maximizing our expected reward. Here, we present a new explicit characterization of the value function as a solution of a quadratic FBSDE with a finite stopping time almost surely.

Let $Q_t = (x_t, y_t)$ be a non-degenerate, bidimensional, weakly coupled, geometric Brownian motion given by

$$\begin{aligned} dx_t &= \mu x_t dt + (\alpha\sqrt{y_t} + \sigma)x_t dB_t^1; \quad x(0) = x, \\ dy_t &= \theta(y_t)dt + \beta(y_t)dB_t^2; \quad y(0) = y, \end{aligned} \tag{2.1}$$

initially starting at $(x, y) \in \mathbb{R}_+^2$, where μ,α and σ are some fixed constants, $\theta(.)$, $\beta(.)$ are Borel measurable functions, and B_t^1, B_t^2 are independent Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We shall consider the following optimal stopping problem:

PROBLEM 2.1. Find a stopping time τ^* and a value function $\Phi(x, y)$, if they exist, such that

$$\Phi(x, y) = \sup_{\tau} \mathbb{E}^{x,y}\left[x_\tau - \int_0^\tau c(y_s)ds - H(y_\tau)\right], \tag{2.2}$$

where the supremum is taken over all finite stopping times, $c(.)$ and $H(.)$ are positive continuous functions such that $H(.)$ is bounded.

Throughout the note, we shall assume the following:

(H2.1) $\theta(.)$ and $\beta(.)$ are measurable functions on $(0, \infty)$ and there exists $K > 0$ such that for $y, z \in \mathbb{R}_+$,

$$(\theta(y) - \theta(z)) + (\beta(y) - \beta(z)) \leq K(y - z), \quad \theta^2(y) + \beta^2(y) \leq K^2(1 + y^2).$$

3. Main results

It can be shown (see Makasu (2008)) that the optimal stopping boundary for problem (2.1) and (2.2) solves the second-order nonlinear ordinary differential equation

$$\frac{1}{2}\beta^2(y)\psi''(y) + \theta(y)\psi'(y) + \left\{\frac{1}{2}(\alpha\sqrt{y} + \sigma)^2 - \mu - c(y)\right\}\psi(y) = \frac{1}{2}\beta^2(y)\frac{\psi'(y)^2}{\psi(y)}, \tag{3.1}$$

on the open interval $(0, \infty)$, subject to the terminal boundary condition

$$\psi(y) = H(y). \tag{3.2}$$

At this point, it is natural and interesting to ask about the probabilistic interpretation (see also Peng (1991)) of Eqs. (3.1) and (3.2), which is indeed the essence of the next assertion. The proof is essentially a consequence of using Ito's formula, Eqs. (3.1) and (3.2). For this reason, we shall omit the details.

LEMMA 3.1. (Probabilistic interpretation)

Let y_t be an arbitrary diffusion process given in (2.1) for all $t \geq 0$. Suppose that $\psi(.) \in C^2(0,\infty)$, then $\psi(y)$ admits the probabilistic interpretation

$$\psi(y) = \mathbb{E}^y p_0, \tag{3.3}$$

where (y_t, p_t, q_t) uniquely solves a weakly coupled quadratic FBSDE

$$\begin{aligned} y_t &= y + \int_0^{t\wedge\tau} \theta(y_s)ds + \int_0^{t\wedge\tau} \beta(y_s)dB_s^2, \\ p_t &= H(y_\tau) + \int_{t\wedge\tau}^{\tau} \left\{ \left(\frac{1}{2}(\alpha\sqrt{y_s}+\sigma)^2 - \mu - c(y_s) \right) p_s - \frac{1}{2} I_{p_s\neq 0} \frac{q_s^2}{p_s} \right\} ds - \int_{t\wedge\tau}^{\tau} q_s dB_s^2, \end{aligned} \tag{3.4}$$

with a finite stopping time almost surely.

Our main result is stated in the next assertion:

PROPOSITION 3.1. *Let $\mathbb{E}[H(y_\tau)]^2 < \infty$ and $Q_t = (x_t, y_t)$ be a non-degenerate, diffusion process given by (2.1). Then, the optimal stopping problem*

$$\Phi(x,y) = \sup_{\tau} \mathbb{E}^{x,y} \left[x_\tau - \int_0^\tau c(y_s)ds - H(y_\tau) \right],$$

is solved by the value function

$$\Phi(x,y) = \log\left(\frac{\mathbb{E}^y p_0}{x} \right), \tag{3.5}$$

and the optimal stopping time

$$\tau^* = \inf\{t \geq 0 : x_t \geq \mathbb{E}^y p_0\}, \tag{3.6}$$

where (y_t, p_t, q_t) is a unique solution of the quadratic FBSDE in (3.4) with τ^* given by (3.6).

REMARK 3.1. Notice that in the special case when $Q_t = (x_t, y_t)$ is a bidimensional geometric Brownian motion and $c(y) \equiv c$, the results in (3.5) and (3.6) give a new explicit characterization of the optimal stopping problem treated in the author's paper (see Makasu (2008)).

In the next section, we shall now state and prove two lemmas which play a crucial role in the proof of our main result. We first introduce the following appropriate function spaces. Denote

$$\mathcal{U}^2 = \left\{ g(t,\omega) : g(t,\omega) \text{ is } \mathcal{F}_t\text{-adapted real-valued such that } \mathbb{E}\int_0^\tau g^2(s,\omega)ds < \infty \right\},$$

and similarly $\mathcal{V}^2$, and

$$L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}) \;=\; \Big\{\xi : \xi \text{ is an } \mathcal{F}_\tau\text{-measurable random variable such that } \mathbb{E}\xi^2 < \infty \Big\},$$

where τ is a finite stopping time almost surely.

DEFINITION 3.1. *A triple of $\mathcal{F}_t$-adapted processes $(y(.), p(.), q(.))$ is said to be a solution of the FBSDE (3.4), iff $(y(.), p(.), q(.)) \in \mathcal{U}^2 \times \mathcal{U}^2 \times \mathcal{V}^2$ and it satisfies (3.4).*

We shall also assume the following:

(H3.2) $H(y_\tau) \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P})$ for each y.

4. Proof of the main result

The next assertion concerns an a priori estimate for the solution of (3.4). Such an estimate allows one to state and prove an existence result for the BSDE in (3.5).

LEMMA 4.1. (**A priori estimate**.) *Assume that (H2.1), (H3.2) hold and $\mathbb{E}[t \wedge \tau] < \infty$ for all $t \geq 0$. If $(y_t, p_t, q_t) \in \mathcal{U}^2 \times \mathcal{U}^2 \times \mathcal{V}^2$ is a solution of the FBSDE (3.4), then*

$$\begin{aligned} &\mathbb{E}\left\{ y_t^2 + p_t^2 - k_1 \int_0^{t\wedge\tau} y_s^2 ds + \int_{t\wedge\tau}^{\tau} \left(k_3 y_s^2 - k_2\right) p_s^2 ds - 2\mathbb{E}\int_{t\wedge\tau}^{\tau} q_s^2 ds \right\} \\ \leq\;& y^2(0) + \mathbb{E}\left\{ H^2(y_\tau) + K^2 \int_0^{t\wedge\tau} ds \right\}, \end{aligned}$$

for all $t \geq 0$, where $k_1 = 1 + K^2$, $k_2 = (\alpha^2/2 + \sigma^2 + \alpha\sigma) - 2c(y_s) - 2\mu$ and $k_3 = \alpha(\alpha/2 + \sigma)$.

Proof. Applying Ito's formula to y_t^2, it follows that

$$y_t^2 = y^2(0) + \int_0^{t\wedge\tau} 2y_s \theta(y_s) ds + \int_0^{t\wedge\tau} 2y_s \beta(y_s) dB_s^2 + \int_0^{t\wedge\tau} \beta^2(y_s) ds.$$

Now taking the expectation and using (H2.1) and Young's inequality, we have

$$\mathbb{E}y_t^2 \leq y^2(0) + K^2 \mathbb{E}\int_0^{t\wedge\tau} ds + (1 + K^2)\mathbb{E}\int_0^{t\wedge\tau} y_s^2 ds. \tag{4.1}$$

Similarly, applying Ito's formula to p_t^2 and using Young's inequality, it follows that

$$\begin{aligned} \mathbb{E}p_t^2 \;\leq\;& \mathbb{E}\left\{ H^2(y_\tau) + \int_{t\wedge\tau}^{\tau} \left(2c(y_s) + 2\mu - (\frac{\alpha^2}{2} + \sigma^2 + \alpha\sigma) - \alpha\left(\frac{\alpha}{2} + \sigma\right) y_s^2 \right) p_s^2 ds \right\} \\ & + 2\mathbb{E}\int_{t\wedge\tau}^{\tau} q_s^2 ds. \end{aligned} \tag{4.2}$$

The desired result now follows immediately from inequalities (4.1) and (4.2). □

In the next assertion, we state and prove:

LEMMA 4.2. (**Uniqueness of solution.**) *Assume that (H2.1) and (H3.2) hold, then the FBSDE (3.4) has at most one solution in* $\mathcal{U}^2 \times \mathcal{U}^2 \times \mathcal{V}^2$.

Proof. Let (y_t^i, p_t^i, q_t^i), $i = 1, 2$, be two solutions of (3.4). Denote $\widehat{Y}_t = y_t^1 - y_t^2$, $\widehat{P}_t = p_t^1 - p_t^2$, $\widehat{Q}_t = q_t^1 - q_t^2$, $\theta(\widehat{Y}_t) = \theta(y_t^1) - \theta(y_t^2)$ and $\beta(\widehat{Y}_t) = \beta(y_t^1) - \beta(y_t^2)$. Assume that (H2.1) holds. Applying Ito's formula to $\widehat{Y}_s e^{\lambda \widehat{P}_s^2}$, it follows that

$$
\begin{aligned}
\mathbb{E}\left(\widehat{Y}_\tau e^{\lambda \widehat{H}^2(\widehat{Y}_\tau)}\right) \quad = \quad & \widehat{Y}(0) e^{\lambda \widehat{P}^2(0)} + \mathbb{E}\left\{\int_0^\tau \left(\theta(\widehat{Y}_t) + 2\lambda \widehat{P}_t \widehat{Q}_t \beta(\widehat{Y}_t) + 2\lambda \widehat{Y}_t \widehat{Q}_t^2\right) e^{\lambda \widehat{P}_t^2} dt\right\} \\
& + \mathbb{E}\left\{\int_0^\tau \left(\lambda[2\mu - \sigma^2] \widehat{Y}_t \widehat{P}_t^2 + 2\lambda^2 \widehat{Y}_t \widehat{Q}_t^2 \widehat{P}_t^2 + 2\lambda c(\widehat{Y}_t) \widehat{Y}_t \widehat{P}_t^2\right) e^{\lambda \widehat{P}_t^2} dt\right\} \\
& - \mathbb{E}\left\{\int_0^\tau \left(\lambda \alpha^2 \widehat{Y}_t^2 \widehat{P}_t^2 + 2\lambda \alpha \sigma \widehat{Y}_t^{3/2} \widehat{P}_t^2\right) e^{\lambda \widehat{P}_t^2} dt\right\} \\
\leq \quad & \widehat{Y}(0) e^{\lambda \widehat{P}^2(0)} + \mathbb{E}\left\{\int_0^\tau \left(K \widehat{Y}_t + 2\lambda K \widehat{P}_t \widehat{Q}_t \widehat{Y}_t + 2\lambda \widehat{Y}_t \widehat{Q}_t^2\right) e^{\lambda \widehat{P}_t^2} dt\right\} \\
& + \mathbb{E}\left\{\int_0^\tau \left(\lambda[2\mu - \sigma^2] \widehat{Y}_t \widehat{P}_t^2 + 2\lambda^2 \widehat{Y}_t \widehat{Q}_t^2 \widehat{P}_t^2 + 2\lambda \widehat{Y}_t c(\widehat{Y}_t) \widehat{P}_t^2\right) e^{\lambda \widehat{P}_t^2} dt\right\} \\
& - \mathbb{E}\left\{\int_0^\tau \left(\lambda \alpha^2 \widehat{Y}_t^2 \widehat{P}_t^2 + 2\lambda \alpha \sigma \widehat{Y}_t^{3/2} \widehat{P}_t^2\right) e^{\lambda \widehat{P}_t^2} dt\right\} \\
\leq \quad & \widehat{Y}(0) e^{\lambda \widehat{P}^2(0)} + \frac{K}{2} \mathbb{E} \int_0^\tau e^{\lambda \widehat{P}_t^2} dt + \mathbb{E}\left\{\int_0^\tau \left(\lambda \widehat{Q}_t^2 + \lambda \gamma_0 \widehat{P}_t^2 \widehat{Q}_t^2 + \lambda \frac{\gamma_1}{2} \widehat{P}_t^2\right) e^{\lambda \widehat{P}_t^2} dt\right\} \\
& + \mathbb{E}\left\{\int_0^\tau \left(K\left(\frac{1}{2} + \lambda\right) + \lambda \gamma_2 \widehat{P}_t^2 + \lambda \widehat{Q}_t^2 + \lambda^2 \widehat{Q}_t^2 \widehat{P}_t^2\right) \widehat{Y}_t^2 e^{\lambda \widehat{P}_t^2} dt\right\}, \qquad (4.3)
\end{aligned}
$$

where the last inequality follows from using Young's inequality, $\gamma_0 = K + \lambda$, $\gamma_1 = 2\mu - \sigma^2 - \alpha\sigma > 0$ and $\gamma_2 = \mu - \sigma^2/2 - \alpha^2 - (3/2)\alpha\sigma + 2$.

From the uniqueness of the FSDE in (3.4) and by choice of $\lambda \leq -K$, we deduce that the third term in the right hand side of the last inequality is negative. Hence, $p_t^1 = p_t^2$ and $q_t^1 = q_t^2$ for all $t \in [0, \tau]$. This completes the proof. $\square$

REMARK 4.1. Notice that in the above proof we apply Ito's formula to $e^{\lambda \widehat{P}_t^2} \widehat{Y}_t$ not $\widehat{Y}_t \widehat{P}_t$. This is one of the main differences between the known uniqueness results on FBSDEs with stopping time and ours.

REMARK 4.2. Finally, it would be interesting to extend the present results to a general class of FBSDEs with jumps (see Yin and Situ (2003), for instance), under our assumptions. The details will be treated elsewhere.

5. Conclusion

In this note, under certain conditions, we have proved an a priori estimate and a uniqueness result for a quadratic FBSDE with a stopping time associated with a bidimensional optimal stopping problem. The present result gives a new explicit characterization of the value function in form of a solution of the quadratic FBSDE with stopping time. This extends a recent result of the author.

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