On a multiple sequential test

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Abstract. We consider a multiple testing problem based on an i.i.d. sample of K-dimensional observations. We want to test whether at least one of the unknown means is positive. We propose a sequential test which is of the nature of a multiple sequential probability ratio test. We asymptotically analyse the expected sample size and compare it to the sample sizes which arise when one looks at effects separately.
Keywords. Sequential test, multiple hypotheses, expected sample size.

Let $\{X_n^{(1)}\}$, ... , $\{X_n^{(K)}\}$ be sequences (not necessarily independent) consisting of i.i.d. random variables within each sequence. Denote $S_n^{(i)} = X_1^{(i)} + \ldots + X_n^{(i)}$, $n \geq 1$, and for arbitrary $b > 0$

$$V_b^{(i)} = \inf\{n \geq 1 : \; S_n^{(i)} \geq b\}, \quad i = 1, \ldots, K.$$

using $\inf \varnothing = \infty$. Denote

$$V_b = \min_{1 \leq i \leq K} V_b^{(i)}.$$

It is our aim to investigate the expected sample size $\mathbf{E}V_b$ in relation to $\min\limits_{1 \leq i \leq K} EV_b^{(i)}$.

This problem arises, for example, in clinical trials. Suppose first that $K = 1$. After applying a likelihood ratio principle and a suitable normalization, we come to a special random walk which has non-positive drift under hypothesis H_1 (no harmful effect of a drug) and positive drift under H_2 (that corresponds to a harmful effects of a drug). The decision for H_2 is taken when our random walk achieves a given level b before a certain time m, and validity of H_2 should be detected as soon as possible. This problem was treated by Irle and Lotov (2004). Assume now that we monitor $K > 1$ different harmful effects of a drug so we could consider not one but $K > 1$ sequences of observations and, correspondingly, K random walks simultaneously. In this case, the decision for H_2 could be taken when at least one trajectory achieves the level b before time m. Under this setting, how much is the saving in sample size in comparison to the case $K = 1$?

It is clear that $\mathbf{E}V_b \leq \mathbf{E}V_b^{(i)}$ for each $i = 1, \ldots, K$, so $\mathbf{E}V_b \leq \min\limits_i \mathbf{E}V_b^{(i)}$. We assume that, for some $\delta \geqslant 0$,

$$\mathbf{E}\,|\,X_1^{(i)}|^{2+\delta} < \infty, \; i = 1, \ldots, K, \text{ and set}$$

$$\mathbf{E}\; X_1^{(i)} = \mu_i.$$

Suppose that

$$\mu_1, \ldots, \mu_l > 0, \; \mu_{l+1}, \ldots, \mu_K \leq 0$$

and $\mu_1 = \max\limits_j \mu_j$. It is well-known that, as $b \to \infty$,

$$\mathbf{E}\,V_b^{(i)} = \frac{b}{\mu_i}\,(1 + o\,(1)), \;\; i = 1, \ldots, l, \;\; \mathbf{E}\,V_b^{(i)} = \infty, \, i = l+1, \ldots, K.$$

It is our aim to compare $\mathbf{E}\,V_b^{(1)}$, the asymptotically smallest value, with $\mathbf{E}\,V_b$, hence investigate the quantity

$$\Delta_b = \mathbf{E}\,V_b^{(1)} - \mathbf{E}\,V_b.$$

We obtain the following result:

Suppose that $\mu_1 > \mu_i,\ 1 < i \leq l$ so that there is a unique dominant mean μ_1. Then we have

$$\Delta_b = O(b^{-\frac{\delta}{2}}), \quad b \to \infty.$$

Suppose that $\mu_1 = \cdots = \mu_j, \quad 2 \leq j \leq l,$ and $\mu_1 > \mu_i$ for $j+1 \leq i \leq l$, so there are two or more dominating effects. Then we have

$$\Delta_b = C\sqrt{b} + o(\sqrt{b}), \quad b \to \infty,$$

for some constant $C > 0$ which is calculated in an explicit form.

Hence looking at effects simultaneously may lead to a noticable saving in sample size in comparison to looking at effects separately.

References

Irle, A. and Lotov, V. (2004). A Nonsymmetric Sequential Test. *Metrika*, **59**, 137–146.