Nonparametric Repeated Significance Tests with Random Sample Size

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Abstract. In this article recent results are presented for nonparametric repeated significance tests for distributions with heavy tails. Of special interest are repeated significance tests with a random target sample size. Functional central limit theorems used in implementation of these tests are discussed as well. Current research initiatives and open problems are also stated.

Keywords. functional central limit theorems, heavy tail distributions, random target sample size, repeated significance tests.

1 Introduction

In this article recent results are reviewed for repeated significance tests for distributions with heavy tails. Repeated significance tests have been introduced in Armitage (1958) and since then have been investigated extensively. Recent developments in the theory and applications of repeated significance tests have been reported among others in: Glaz and Pozdnyakov (2005), Gu and Lai (1998), Ho (1998), Jennison and Turnbull (2000), Lee, Kim and Tsiatis (1996), Lerche (1986), Sen (2002), Siegmund (1985), Takahashi (1990) and Whitehead (1997). Nonparametric repeated significance tests have been discussed in Glaz and Pozdnyakov (2005) and Sen (1981 and 2002).

Pozdnyakov and Glaz (2007) introduced a repeated significance test with a square root boundary that has a random target sample size. The random target sample size is related to the gross rate of the sample variances of the sequence of test statistics used by the repeated significance test. The advantage in using a repeated significance test with an adaptive target sample size is that its implementation does assume the knowledge of the asymptotic tail behavior of the distribution of the observed data, a crucial condition in developing a robust repeated significance test for distributions with heavy tails (Glaz and Pozdnyakov 2005).

The article is organized as follows. In Section 2, a robust repeated significance test derived in Glaz and Pozdnyakov (2005) is reviewed. In Section 3, a repeated significance test with a random sample size, introduced in Pozdnyakov and Glaz is presented. The functional central limit theorems based on which these test are derived are discussed as well. In Section 4, we present some current research initiatives and open problems.

2 A Robust nonparametric repeated significance test

Let $X_1, X_2, ..., X_n, ...$ be a sequence of independent and identically distributed (iid) observations from a continuous distribution F with median $-\infty < \theta < \infty$. We want to test $H_0 : \theta = 0$ vs $H_a : \theta \neq 0$ with a nonparametric sequential procedure using at most N observations. For the case, $\sigma^2 = Var(X_1) < \infty$, Sen (1981) developed the following repeated significance test presented.

Let $S_n = \sum_{i=1}^{n} X_i$ and

$$\tau = \min\left\{n_0 \leq n \leq N; |S_n| \geq b\sigma\sqrt{n}\right\},$$

a stopping time, n_0 is the initial sample size, N is the target sample size and $b > 0$ is a constant. The repeated significance test stops and rejects H_0 if and only if $\tau \leq N$. The power function of this test is given by:

$$\begin{aligned}\beta(\theta) &= P_\theta(\tau \leq N) = 1 - P_\theta(\tau > N) \\ &= 1 - P_\theta\left(|S_n| < b\sigma\sqrt{n}; n_0 \leq n \leq N\right).\end{aligned}$$

The following functional central limit theorem due to Donsker (see Billingsley 1995, pp. 520) plays an important role in the implementation of the repeated significance test. Let $\{X, X_i\}_{i\geq 1}$ be iid random variables, $EX = \mu$, $VarX = \sigma^2$ and $S_n = X_1 + X_2 + ... + X_n$. If $S_n(t)$ is the linear interpolation between points

$$\Big(0,0\Big), \Big(\frac{1}{n}, \frac{S_1-\mu}{\sigma\sqrt{n}}\Big), ..., \Big(1, \frac{S_n - n\mu}{\sigma\sqrt{n}}\Big)$$

then

$$S_n(t) \xrightarrow{d} W$$

in the sense $\mathcal{C}[0,1]$ with uniform metric ρ where W is standard Brownian motion on $[0,1]$.

For a specified significance level $\alpha > 0$, if $n_0/N \to t_0$ as $N \to \infty$, it follows from Donsker's Theorem that

$$\max\left\{\frac{|S_n|}{\sigma\sqrt{n}}; n_0 \leq n \leq N\right\} \xrightarrow{d} \sup\left\{\frac{W(t)}{\sqrt{t}}; t_0 \leq t \leq 1\right\}$$

and

$$\beta(0) = P_0\left(\max\left\{\frac{|S_n|}{\sigma\sqrt{n}}; n_0 \leq n \leq N\right\} \geq b\right) \to \alpha,$$

where $W(t)$ is a standard Brownian motion on the interval $[0,1]$ and $b = b_{t_0}(\alpha)$ is the constant that characterizes the continuation region, given by the square root boundary, corresponding to the prescribed significance level α.The critical values $b_{t_0}(\alpha)$ for different choices of α and t_0 can obtained from DeLong (1981). If σ is unknown, one can replace it by the sample standard deviation since it converges almost surely to σ (Sen 1981).

Let $X, X_1, X_2, ..., X_n, ...$ be a sequence of iid observations from a continuous symmetric distribution F with median $-\infty < \theta < \infty$. For F in the class of heavy tail distributions with an infinite variance and possibly no mean, Glaz and Pozdnyakov (2005) derived a repeated significance test testing $H_0 : \theta = 0$ vs $H_a : \theta \neq 0$. This repeated significance test is constructed as follows. Let $\{b_n\}_{n\geq 1}$ be an increasing sequence of positive numbers such that

$$n\mathbf{P}\big(|X| > b_n\big) \sim \gamma_n \nearrow \infty.$$

Denote by S_n^*

$$S_n^* = \sum_{i=1}^{n} X_i \mathrm{I}_{(|X_i|\leq b_n)}, \tag{1}$$

the partial sums of a truncated sequence of observations and let $B_n = Var(S_n^*)$. Let

$$\tau = \min\left\{n_0 \leq n \leq N; |S_n^*| \geq b_n\sqrt{A_n}\right\}$$

be a stopping time, where

$$A_n = \sum_{i=1}^{n} X_i^2 \mathrm{I}_{|X_i|\leq b_n} - \frac{S_n^{*2}}{\sum_{i=1}^{n} \mathrm{I}_{|X_i|\leq b_n}}, \tag{2}$$

is a sequence of sample estimators of B_n, n_0 and N are the initial and the target sample size, respectively. The repeated significance test stops and rejects H_0 if and only if $\tau \leq N$. The power function of this test is given by

$$\begin{aligned}\beta(\theta) &= P_\theta(\tau \leq N) = 1 - P_\theta(\tau > N) \\ &= 1 - P_\theta\left(|S_n^*| < b_n\sqrt{A_n}; n_0 \leq n \leq N\right).\end{aligned} \tag{3}$$

Since the sequence of partial sums of truncated random variables, S_n^*, is not a process with independent increments, the classical Donsker functional central limit theorem cannot be used. However, in the case of symmetric distributions, $\{S_n^*\}$ is a martingale. The following analog of Donsker's theorem plays an important role in the implementation of the repeated significance test given above.

Theorem 1. *(Pozdnyakov 2003) If the random variable X belongs to the Feller class:*

$$\limsup_{t\to\infty} \frac{t^2 P(|X|>t)}{E\big(X^2 \mathrm{I}_{|X|\le t}\big)} < \infty,$$

the average number of the excluded observations

$$nP\big(|X| > b_n\big) \sim \gamma_n \nearrow \infty,$$

and $B_n/B_{n+1} \to 1$ then $S_n^*(t) \xrightarrow{d} W$ in the sense $(\mathcal{C}[0,1], \rho)$, where $S_n^*(t)$ is the linear interpolation between points

$$\Big(0,0\Big), \Big(\frac{B_1}{B_n}, \frac{S_1^*}{\sqrt{B_n}}\Big), ..., \Big(1, \frac{S_n^*}{\sqrt{B_n}}\Big).$$

Glaz and Pozdnyakov (2005) show that for the problem at hand

$$\frac{A_n}{B_n} \to 1 \text{ a.s..}$$

Therefore, under H_0 if $B_{n_0}/B_N \to t_0$ and $N \to \infty$, then

$$\max\left\{\frac{|S_n|}{\sqrt{A_n}}; n_0 \le n \le N\right\} \xrightarrow{d} \sup\left\{\frac{W(t)}{\sqrt{t}}; t_0 \le t \le 1\right\}$$

and consequently,

$$\beta(0) = P_0\left(\max_{n_0\le n\le N}\left\{\frac{|S_n^*|}{\sqrt{A_n}}\right\} \ge b_n\right) \to \alpha,$$

where the constant $b_n = b_n(\alpha)$ is the critical value that determines the continuation region of the repeated significance test.

Glaz and Pozdnyakov (2005) derived an approximation for $b_n(\alpha)$ for the class of symmetric stable continuous distributions with exponent $0 < \gamma < 2$, i.e.

$$E\left(X^2 \mathrm{I}_{(|X|\le t)}\right) \sim t^{2-\gamma} L(t),$$

where $L(t)$ is a slowly varying function. Based on the invariance principle in Pozdnyakov (2003), it follows that for $b_n = bn^\delta$, $0 < \gamma < 2$, $0 < \delta < 1/2$, $n_0, N \to \infty$ and $n_0/N \to c$, $0 < c < 1$

$$\max\left\{\frac{|S_n^*|}{\sqrt{A_n}}; n_0 \le n \le N\right\} \xrightarrow{d} \sup_{[c^{1+(2-\gamma)\delta},1]} \frac{|W(t)|}{\sqrt{t}}. \tag{4}$$

The constant $b_n(\alpha)$ can be approximated by $b_{t_0}(\alpha)$ by solving

$$P\left(\sup_{[c^{1+(2-\gamma)\delta},1]}\left\{\frac{|W(t)|}{\sqrt{t}}\right\} \ge b_{t_0}(\alpha)\right) = \alpha, \tag{5}$$

using the approach in De Long (1981). Numerical results in Glaz and Pozdnyakov (2005) show that the approximations for the critical value $b_n(\alpha)$ are good.

3 Repeated significance tests with random stopping time

Let $X, X_1, X_2, ..., X_n, ...$ be a sequence of iid observations from a continuous symmetric distribution F in the class of heavy tail distributions with an infinite variance and possibly no mean. Let $-\infty < \theta < \infty$, be the median F. For testing $H_0 : \theta = 0$ vs $H_a : \theta \ne 0$, Pozdnyakov and Glaz (2007) derived a repeated significance test with random target sample size. Let A_n be a sample variance of S_n^*, given in (2) and (1), respectively. Define a stopping time $\mathcal{N}$ by

$$\mathcal{N} = \inf\{k \ge n_0 : \frac{A_k}{A_{n_0}} \ge \frac{1}{t_0}\}, \tag{6}$$

where $0 < t_0 < 1$ is a design parameter. A repeated significance test with random target sample size is defined as follows. Let

$$\tau = \inf\left\{k \geq n_0 : |S_k^*| \geq b\sqrt{A_k}\right\}$$

be a stopping time, where n_0 is the initial sample size, and $\mathcal{N}$ is the random target sample size defined in (6). The repeated significance test stops and rejects H_0 if and only if $\tau \leq \mathcal{N}$. Therefore, $\tau \wedge \mathcal{N}$ is the stopping time associated with this repeated significance test. The following result from Pozdnyakov and Glaz (2007) plays a key role in implementing the repeated significance test with random target sample size.

Theorem 2. *Assume that the functional central limit theorem for the sequence $\{S_n^*\}$ holds, and the sequence of $B_n = Var(S_n^*)$ satisfies: $B_n \nearrow \infty$ and $B_n/B_{n-1} \to 1$, as $n \nearrow \infty$. If*

$$\frac{A_n}{B_n} \to 1 \quad a.s., \tag{7}$$

then

$$P\left(\max_{n_0 \leq k \leq \mathcal{N}} \left|\frac{S_k^*}{\sqrt{A_k}}\right| > b\right) \longrightarrow \alpha(t_0, b) \text{ as } n_0 \to \infty. \tag{8}$$

For observations modeled by a distribution in the Feller class such as the Cauchy distribution, Theorem 1 implies that a functional central limit theorem holds and the repeated significance test presented above can be carried out. Pozdnyakov and Glaz (2007) present numerical results indicating that this sequential test performs well. The advantage of using the repeated significance test with random target sample size over the one investigated in Glaz and Pozdnyakov (2005) is that for the design of the test we do not need to specify the asymptotic tail behavior of the heavy tail distribution. For power calculations one still needs to specify the asymptotic tail behavior.

4 Concluding remarks

Recently, Guerriero et. al. (2009) derived a repeated significance test with a random target sample size that is controlled by the total available resources to carry out the sequential testing procedure. This test was applied to a decentralized sequential detection problem in a sensor communication network with communication constraints.

Glaz and Kenyon (1995) presented an approach used in developing median unbiased confidence intervals after the completion of a sequential testing procedure. It will be of interest to develop the algorithms needed to implement this methodology to repeated significance tests with random target sample size.

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