Sequential Estimation of The Mean of a Normal Distribution Under LINEX Loss

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Abstract. The present paper studies a sequential stopping rule for estimating the mean of a normal distribution, under LINEX loss, when the variance is unknown. The exact distribution of the stopping variable is derived, as well as exact formulae for the expected estimator of the mean, and its risk functional. The results of the present paper are different from those of Chattopadhyay et al (2005) or those of Takada (2006).

Keywords. Bounded risk estimation, Exact distributions, LINEX Loss, Normal Mean, Sequential Procedures.

1 Introduction

A sequential application under a LINEX loss function was first developed by Chattopadhyay (1998). Under this loss function, Takada (2000) introduced a sequential minimum risk point estimation problem for a normal mean whereas Takada (2001) gave a Bayesian formulation for sequentially estimating a Poisson mean. Takada and Nagao (2004) introduced extensions of such estimation problems under the same loss function assuming a multivariate normal distribution. Chattopadhyay et al. (2005) gave a purely sequential sampling scheme followed by batch sampling in order to address a prediction problem in linear regression under the same loss function.

The present article derives the asymptotic properties, the exact distributions of stopping times, and the associated functionals for estimating the mean in a normal distribution under a LINEX loss function. The purely sequential bounded risk procedures proposed in this paper in the light of Robbins (1959) are different from those of Takada (2006) and Chattopadhyay (1998). The main reason for this difference stems from the fact that their earlier stopping rules and those that we have introduced here are different.

2 Fixed Sample Risk Evaluations

Let $X_1, X_2, \ldots$ be *independent and identically distributed* (i.i.d.) random variables, having a $N(\mu, \sigma^2)$ distribution with $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. We assume that both parameters are unknown. The objective is to estimate the mean μ under the LINEX loss function:

$$L_n(\hat{\mu}_n, \mu) = \exp\{a(\hat{\mu}_n - \mu)\} - a(\hat{\mu}_n - \mu) - 1, \tag{1}$$

where $\hat{\mu}_n \equiv \hat{\mu}_n(X_1, \ldots, X_n)$ is an estimator of μ constructed from a random sample $X_1, \ldots, X_n$ of fixed size n where $a \neq 0$ is held fixed.

Let us denote $\theta = (\mu, \sigma^2)$. When σ^2 is known, an estimator of μ, when $\bar{X}_n$ is the sample mean, is

$$\hat{\mu}_n = \bar{X}_n - \frac{a\sigma^2}{2n}. \tag{2}$$

The corresponding LINEX risk value is

$$E_\theta\left\{L\left(\bar{X}_n - \frac{1}{2}\frac{a\sigma^2}{n}, \mu\right)\right\} = \frac{a^2\sigma^2}{2n}. \tag{3}$$

For a given risk-bound ω, $0 < \omega < \infty$, if σ^2 were known, the risk associated with $\hat{\mu}_n$ will be smaller than ω when n is the smallest integer $\geq n_\omega$, where

$$n_\omega \equiv n_\omega(\sigma^2) = \frac{a^2\sigma^2}{2\omega}. \tag{4}$$

Since σ^2 is unknown, we use the following estimator

$$\tilde{\mu}_n = \bar{X}_n - \frac{a}{2}\frac{S_n^2}{n}, \tag{5}$$

where S_n^2 is the sample variance. The bias of this estimator is

$$\text{Bias}(\tilde{\mu}_n) = -\frac{a}{2}\frac{\sigma^2}{n}. \tag{6}$$

Notice that $\tilde{\mu}_n$ is translation and scale equivariant. The LINEX risk of $\tilde{\mu}_n$ is

$$\begin{aligned} R(\tilde{\mu}_n) &= E_\theta\left\{\exp\left(a\left(\bar{X}_n - \mu - \frac{aS_n^2}{2n}\right)\right)\right\} - E\left\{a\left(\bar{X}_n - \mu - \frac{aS_n^2}{2n}\right)\right\} - 1 \\ &= E_\theta\{e^{a(\bar{X}_n-\mu)}\}E_\theta\left\{\exp\left(-\frac{a^2S_n^2}{2n}\right)\right\} + \frac{a^2\sigma^2}{2n} - 1 \\ &= \left(1 + \frac{a^2\sigma^2}{n(n-1)}\right)^{-(n-1)/2}\exp\left(\frac{a^2\sigma^2}{2n}\right) + \frac{a^2\sigma^2}{2n} - 1. \end{aligned} \tag{7}$$

3 Sequential Sampling

3.1 The Stopping Variables

We begin with $m_0(= 2k+1)$ initial observations $X_1, \ldots, X_{m_0}$ and then continue sampling, if needed, by observing a pair of observations at a time. Let $M \equiv M_{m_0}$ denote the final number of pairs observed, $M_{m_0} \geq k$ and the final number of recorded observations $N \equiv N_{m_0}(= 2M_{m_0}+1)$ where the stopping variable M_{m_0} is formulated as follows:

$$M_{m_0} = \min\left\{m \geq k : 2m+1 \geq \frac{a^2}{2\omega}S_{2m+1}^2\right\}. \tag{8}$$

Finally, μ is estimated by $\tilde{\mu}_{N_{m_0}} = \bar{X}_{N_{m_0}} - \dfrac{a}{2N_{m_0}}S_{N_{m_0}}^2$.

The stopping rule from (8) can be equivalently expressed as

$$M_{m_0} = \min\left\{m \geq k : 2m+1 \geq \frac{a^2\sigma^2}{2\omega m}\sum_{i=1}^{m}T_i\right\}, \tag{9}$$

where $T_1, T_2, \ldots$ denote an i.i.d. sequence of exponentially distributed random variables with mean $\beta = 1$. Obviously, $\sum_{i=1}^{m}T_i$ and $G(1,m)$ have identical distributions for every fixed $m \geq k$.

As before, let $P(\cdot;\eta)$ and $p(\cdot;\eta)$ respectively denote the c.d.f. and p.m.f. of a Poisson random variable with mean η. Also, let

$$\lambda = 2\omega/(a^2\sigma^2), \quad 0 < \lambda < \infty.$$

According to the Gamma-Poisson relationship we have

$$\begin{aligned} P_{\sigma^2}\{M_{m_0} = k\} &= P\{G(1,k) \leq \lambda k(2k+1)\} \\ &= 1 - P(k-1; \lambda k(2k+1)). \end{aligned} \tag{10}$$

In order to determine the distribution of M_{m_0}, on the set $\{M_{m_0} > k\}$, we introduce the homogeneous Poisson process $\{M(t), t \geq 0\}$ with intensity $\beta = 1$. The distribution of the m^{th} jump point of $M(t)$ is like that of $G(1,m)$. The stopping variable M_{m_0}, defined in (9), is therefore equivalent to the value of M at the first time when $(2M(t)+1)M(t) \geq \lambda^{-1}t$.

We therefore define a boundary $B(t)$ to be the positive root of the quadratic equation $2M^2(t) + M(t) - \lambda^{-1}t = 0$. That is,

$$B(t) = -\frac{1}{4} + \frac{1}{4}\sqrt{1 + \frac{8}{\lambda}t}, \quad t \geq 0. \tag{11}$$

Notice that $B(t)$ is a continuous, increasing concave function, with $B(0) = 0$ and $\lim_{t\to\infty} B(t) = \infty$. We define another stopping time

$$\tilde{T} = \inf\{t \geq t_k : M(t) \geq B(t)\}, \tag{12}$$

where

$$t_k = \lambda k(2k + 1). \tag{13}$$

In general, for all $m \geq k$, let t_m be the root of $B(t) = m$, namely

$$t_m = \lambda m(2m + 1), \quad m \geq k. \tag{14}$$

Since $M(t_k) \sim \text{Poisson}(t_k)$, we have

$$P_{\sigma^2}\{\tilde{T} = t_k\} = 1 - P(k - 1; t_k) = P_{\sigma^2}\{M(\tilde{T}) = k\} = P_{\sigma^2}\{M_{m_0} = k\}. \tag{15}$$

Generally, for all $M_{m_0} \geq k$, M_{m_0} and $M(\tilde{T})$ have the same distribution. Again, $\lfloor B(t) \rfloor$ is the largest integer smaller than $B(t)$. Then, $M(\tilde{T}) = \lfloor B(\tilde{T}) \rfloor + 1$.

3.2 The Distribution and the Mean of M_{m_0}

Define the defective p.d.f.

$$g(i; t) = P_{\sigma^2}\{M_{m_0}(t) = i, \tilde{T} > t\}. \tag{16}$$

Notice that $g(i; t) = 0$ for all $i \geq B(t)$, and

$$P_{\sigma^2}\{\tilde{T} > t\} = \sum_{i=0}^{\lfloor B(t) \rfloor} g(i; t). \tag{17}$$

Theorem 1. *Under sequential sampling (9), for all $m > k$, we have*

$$P_{\sigma^2}\{M_{m_0} = m\} = \sum_{j=0}^{m-2} g(j; t_{m-1})(1 - P(m - 1 - j; t_m - t_{m-1})), \tag{18}$$

and

$$E_{\sigma^2}\{M_{m_0}\} = k + \sum_{j=k}^{\infty} \sum_{i=0}^{j-1} g(i; t_j). \tag{19}$$

3.3 The Expected Value and the LINEX Risk of $\tilde{\mu}_{N_{m_0}}$

Under sequential sampling according to (8), we consider the following estimator of μ at stopping:

$$\tilde{\mu}_{N_{m_0}} = \bar{X}_{N_{m_0}} - \frac{a}{2N_{m_0}} S^2_{N_{m_0}}. \tag{20}$$

In this section, we develop the bias and the LINEX risk of $\tilde{\mu}_{N_{m_0}}$.

Observe that $\bar{X}_n$ and $(S^2_{m_0}, \ldots, S^2_n)$ are independent for all fixed $n \geq m_0$. Now, since $I\{N_{m_0} = n\}$ belongs to the subfield $\sigma(S^2_{m_0}, \ldots, S^2_n)$, we have $E_{\sigma^2}\{\bar{X}_{N_{m_0}}\} = \mu$. Thus the bias of $\tilde{\mu}_{N_{m_0}}$ is $-\frac{a}{2} E_{\sigma^2}\{S^2_{N_{m_0}}/N_{m_0}\}$.

For the proofs of the following two theorems, see Zacks and Mukhopadhyay (2009).

Theorem 2. *Under sequential sampling (8), we have the following expression for the bias of* $\tilde{\mu}_{N_{m_0}}$, *with* $\Delta_l = t_l - t_{l-1}$,

$$\begin{aligned}\text{Bias}(\tilde{\mu}_{N_{m_0}}) = -\frac{a\sigma^2}{2}\Bigg[&\frac{t_k}{k(2k+1)}(1-P(k-1;t_k))\\&+\sum_{l=k+1}^{\infty}\frac{1}{l(2l+1)}\sum_{i=0}^{l-2}g(i;t_{l-1})\\&\times(t_{l-1}(1-P(l-1-i;\Delta_l))+(l-i)(1-P(l-i;\Delta_l)))\Bigg].\end{aligned}\tag{21}$$

Since

$$E_\theta\{\exp(a(\bar{X}_{N_{m_0}}-\mu))\mid N_{m_0}\} = \exp\left(\frac{a^2\sigma^2}{2N_{m_0}}\right).\tag{22}$$

Thus, we can rewrite

$$R(\tilde{\mu}_{N_{m_0}}) = E_{\sigma^2}\left\{\exp\left(-\frac{a^2(S^2_{N_{m_0}}-\sigma^2)}{2N_{m_0}}\right)\right\} + \frac{a^2}{2}E_{\sigma^2}\left\{\frac{S^2_{N_{m_0}}}{N_{m_0}}\right\} - 1.\tag{23}$$

The second term on the right-hand side of (23) is found from (21).

Theorem 3. *Under sequential sampling (8), we have*

$$\begin{aligned}E_{\sigma^2}\left\{\exp\left(-\frac{a^2(S^2_{N_{m_0}}-\sigma^2)}{2N_{m_0}}\right)\right\} &= (1-P(k-1;t_k))\exp\left(-\omega\left(1-\frac{k}{t_k}\right)\right)\\+\sum_{l=k+1}^{\infty}&\exp\left(-\omega\left(\left(1-\frac{1}{l}\right)\left(1-\frac{2}{2l+1}\right)-\frac{l}{t_l}\right)\right)\\\times\sum_{i=0}^{l-2}&g(i;t_{l-1})\left(1+\frac{\omega}{t_l}\right)^{-(l-i)}\left(1-P\left(l-l-i;\Delta_l\left(1+\frac{\omega}{t_l}\right)\right)\right).\end{aligned}\tag{24}$$

3.4 Numerical Example

In Table 1, within each block, the first row provides the exact values of $E\{N_{m_0}\}$, $\text{Bias}(\tilde{\mu}_{N_{m_0}})$ and $R(\tilde{\mu}_{N_{m_0}})$ when we fixed $k = 9$, $a = 5$, $\sigma = 1, 2$ and $\omega = 0.5, 0.1$.

Table 1. Exact operating characteristics of sequential estimator $\tilde{\mu}_{N_{m_0}}$

σ	ω	n_ω	$E\{N_{m_0}\}$	$\text{Bias}(\tilde{\mu}_{N_{m_0}})$	$\text{Risk}(\tilde{\mu}_{N_{m_0}})$
1	0.5	25	25.55	−0.09576	0.54498
1	0.1	125	124.33	−0.01976	0.10424
2	0.5	100	99.28	−0.09848	0.53329
2	0.1	500	499.57	−0.015000	0.100950

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