On Exact and Asymptotic Properties of Two-Stage Estimation of The Normal Mean Under LINEX Loss

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Abstract. In this article, we consider the bounded risk point estimation problem for the mean μ in a $N(\mu, \sigma^2)$ distribution under a LINEX loss function. We have proposed a two-stage procedure with a goal that the associated risk functions approximately fall under a preassigned risk-bound $\omega(> 0)$. Our two-stage procedure and the associated terminal estimators for μ are different from those of Takada (2006) and Chattopadhyay et al. (2005) respectively. We begin by presenting interesting asymptotic properties of the two-stage point estimator for μ followed by the exact distribution of the two-stage stopping time along with some exact properties of the terminal estimator for μ. The exact expressions are illustrated with numerical computations.

Keywords. Asymptotic risk bound, Bounded risk problem, Exact properties, LINEX loss, Normal mean, Sequential procedure, Two-stage procedure.

1 Introduction

The present paper is a short version of a new paper by Zacks and Mukhopadhyay (2009). This paper is a continuation of the study on bounded risk estimation under two-stage and sequential sampling (see Zacks and Mukhopadhyay 2006). The special feature of the present paper is estimation of the mean of a normal distribution, with unknown variance, under a *linear-exponential* (LINEX) loss function, which was introduced by Varian (1975) and featured by Zellner (1986) and Rojo (1987).

Takada (2006) had developed two-stage and three-stage bounded risk estimation problems for a normal mean under the LINEX loss with some second-order asymptotic properties under an assumption that the unknown variance has a known positive lower bound. The cited papers of Takada and Chattopadhyay with their respective colleagues largely discussed various asymptotic properties associated with two-stage, three-stage, and purely sequential estimation procedures.

The present article derives the asymptotic properties, the exact distributions of stopping times, and the associated functionals for estimating the mean in a normal distribution under a LINEX loss function. The two-stage bounded risk procedure proposed in this paper in the light of Stein (1945, 1949) is different from those of Takada (2006) and Chattopadhyay (1998). That is so even though our final estimators of μ look the same as in those papers. The main reason for this difference stems from the fact that their earlier stopping rules and those that we have introduced here are different.

2 Fixed Sample Risk Evaluations

Let $X_1, X_2, \ldots$ be *independent and identically distributed* (i.i.d.) random variables, having a $N(\mu, \sigma^2)$ distribution with $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. We assume that both parameters are unknown. The objective is to estimate the mean μ under the LINEX loss function:

$$L_n(\hat{\mu}_n, \mu) = \exp\{a(\hat{\mu}_n - \mu)\} - a(\hat{\mu}_n - \mu) - 1, \tag{1}$$

where $\hat{\mu}_n \equiv \hat{\mu}_n(X_1, \ldots, X_n)$ is an estimator of μ constructed from a random sample $X_1, \ldots, X_n$ of fixed size n where $a \neq 0$ is held fixed.

Let us denote $\theta = (\mu, \sigma^2)$. When σ^2 is known, Zellner (1986) and Rojo (1987) have shown that an admissible minimax estimator of μ, when $\bar{X}_n$ is the sample mean, is

$$\hat{\mu}_n = \bar{X}_n - \frac{a\sigma^2}{2n}. \tag{2}$$

The corresponding LINEX risk value is

$$E_\theta \left\{ L\left(\bar{X}_n - \frac{1}{2}\frac{a\sigma^2}{n}, \mu \right) \right\} = \frac{a^2\sigma^2}{2n}. \tag{3}$$

For a given risk-bound ω, $0 < \omega < \infty$, if σ^2 were known, the risk associated with $\hat{\mu}_n$ will be smaller than ω when n is the smallest integer $\geq n_\omega$, where

$$n_\omega \equiv n_\omega(\sigma^2) = \frac{a^2\sigma^2}{2\omega}. \tag{4}$$

Since σ^2 is unknown, we use the following estimator

$$\tilde{\mu}_n = \bar{X}_n - \frac{a}{2}\frac{S_n^2}{n}, \tag{5}$$

where S_n^2 is the sample variance. The bias of this estimator is

$$\text{Bias}(\tilde{\mu}_n) = -\frac{a}{2}\frac{\sigma^2}{n}. \tag{6}$$

Notice that $\tilde{\mu}_n$ is translation and scale equivariant. The LINEX risk of $\tilde{\mu}_n$ is

$$\begin{aligned} R(\tilde{\mu}_n) &= E_\theta \left\{ \exp\left(a\left(\bar{X}_n - \mu - \frac{aS_n^2}{2n} \right) \right) \right\} - E\left\{ a\left(\bar{X}_n - \mu - \frac{aS_n^2}{2n} \right) \right\} - 1 \\ &= E_\theta\{e^{a(\bar{X}_n - \mu)}\} E_\theta \left\{ \exp\left(-\frac{a^2 S_n^2}{2n} \right) \right\} + \frac{a^2\sigma^2}{2n} - 1 \\ &= \left(1 + \frac{a^2\sigma^2}{n(n-1)} \right)^{-(n-1)/2} \exp\left(\frac{a^2\sigma^2}{2n} \right) + \frac{a^2\sigma^2}{2n} - 1. \end{aligned} \tag{7}$$

A series expansion of $R(\tilde{\mu}_n)$, in terms of powers of $1/n$, yields

$$R(\tilde{\mu}_n) = \frac{a^2\sigma^2}{2n} + o\left(\frac{1}{n}\right), \quad \text{as } n \to \infty. \tag{8}$$

This coincides with $n_\omega(\sigma^2)$ given by (4). Thus, if $n \geq n_\omega(\sigma^2)$, the LINEX risk of $\tilde{\mu}_n$ will be smaller than $\omega + o\left(\frac{1}{n}\right)$.

3 Two-Stage Sampling

We propose the following two-stage estimation procedure.

Stage 1. Take $m_0 = 2k+1$ initial observations and compute $S_{m_0}^2$ where $k(\geq 1)$ is a fixed integer. Define the stopping variable

$$N_{m_0} = \max\left\{ m_0, \left\lfloor \frac{b_{m_0}a^2}{2\omega} S_{m_0}^2 \right\rfloor + 1 \right\}, \tag{9}$$

where $\lfloor a \rfloor$ = maximal integer smaller than a and $b \equiv b_{m_0}(> 0)$ is a multiplier that is to be appropriately determined. If $N_{m_0} = m_0$, stop sampling and estimate μ by $\tilde{\mu}_{m_0} = \bar{X}_{m_0} - \frac{a}{2m_0} S_{m_0}^2$. If, $N_{m_0} > m_0$, go to Stage 2.

Stage 2. Let $N^* = (N_{m_0} - m_0)^+$. Take additional N^* observations, $X_1^*, \ldots, X_{N^*}^*$. Compute

$$\bar{X}_{N_{m_0}} = \frac{1}{N_{m_0}} \left(\sum_{i=1}^{m_0} X_i + \sum_{j=1}^{N^*} X_j^* \right), S_{N_{m_0}}^2 = \frac{1}{N_{m_0} - 1} \left[\sum_{i=1}^{m_0} X_i^2 + \sum_{j=1}^{N^*} X_j^{*2} - N_{m_0} \bar{X}_{N_{m_0}}^2 \right],$$

and

$$\tilde{\mu}_{N_{m_0}} = \bar{X}_{N_{m_0}} - \frac{a}{2N_{m_0}} S_{N_{m_0}}^2. \tag{10}$$

3.1 Asymptotic Properties Under (9)

The main asymptotic result is given in the following

Theorem 1. *Under two-stage sampling (9), the LINEX risk $R(\tilde{\mu}_{N_{m_0}}) \le \omega + o(n_\omega^{-1})$ if $m_0 > 7$ and*

$$b \equiv b_{m_0} = \frac{m_0 - 1}{m_0 - 3} \tag{11}$$

where n_ω comes from (4).

See Zacks and Mukhopadhyay (2009) for a proof of this theorem.

Remark 1. In defining the two-stage stopping variable in (9), if we had used $\frac{a^2}{2\omega} S_{m_0}^2$ instead of $\frac{b_{m_0} a^2}{2\omega} S_{m_0}^2$, then N_{m_0} would closely mimic the expression of n_ω. But, since σ^2 is unknown, intuitively we should expect to take more than n_ω observations. In other words, the fact that we have found $b \equiv b_{m_0}$ larger than one is consistent with this sentiment.

Remark 2. Our two-stage methodology from (9) used a pilot sample size $m_0 = 2k + 1$ which is an odd integer. Why we do so should become very clear from Subsections 3.2-3.3. Theorem 1 holds whether $m_0 (> 7)$ is odd or even.

3.2 The Distribution of N_{m_0}

Since $m_0 = 2k + 1$, $S_{m_0}^2 \sim \frac{\sigma^2}{2k} \chi^2[2k] \sim \frac{\sigma^2}{k} G(1, k)$, where $G(1, k)$ denotes a gamma random variable, with shape parameter k, and scale parameter 1. Define

$$\lambda_{m_0}(\sigma^2) = \frac{b_{m_0} a^2 \sigma^2}{2\omega}. \tag{12}$$

From the definition of N_{m_0}, we have by the Gamma-Poisson relationship (Kao, 1997, p. 50),

$$P_{\sigma^2}\{N_{m_0} = m_0\} = P\left\{ G(1, k) \le \frac{k m_0}{\lambda_{m_0}(\sigma^2)} \right\} = 1 - P\left(k - 1; \frac{k m_0}{\lambda_{m_0}(\sigma^2)} \right), \tag{13}$$

where $P(\cdot; \eta)$ denotes the cumulative distribution function (c.d.f.) of a Poisson distribution with mean η. Similarly, for $n \ge m_0 + 1$, we can write

$$\begin{aligned} P_{\sigma^2}\{N_{m_0} = n\} &= P\left\{ \frac{k(n-1)}{\lambda_{m_0}(\sigma^2)} < G(1, k) \le \frac{kn}{\lambda_{m_0}(\sigma^2)} \right\} \\ &= P\left(k - 1; \frac{k(n-1)}{\lambda_{m_0}(\sigma^2)} \right) - P\left(k - 1; \frac{kn}{\lambda_{m_0}(\sigma^2)} \right). \end{aligned} \tag{14}$$

Moreover, for $0 < p < 1$, the p-th quantile of N_{m_0} is given by

$$N_{m_0, p} = \min\left(n \ge m_0 : P\left(k - 1; \frac{kn}{\lambda_{m_0}(\sigma^2)} \right) \le 1 - p \right). \tag{15}$$

Also, the expected value of N_{m_0} is given by

$$E_{\sigma^2}\{N_{m_0}\} = m_0 + \sum_{n=m_0}^{\infty} P\left(k-1; \frac{kn}{\lambda_{m_0}(\sigma^2)}\right). \tag{16}$$

In Table 1, we present $Q_1 = N_{m_0,0.25}$, $Me = N_{m_0,0.5}$, $Q_3 = N_{m_0,0.75}$, $E\{N_{m_0}\}$ and $n_\omega(\sigma^2)$ when $a = 5$, $\sigma = 1$, $\omega = 0.5$, 0.1 and $k = 5$ (that is, $m_0 = 11$), 10 (that is, $m_0 = 21$), both by exact calculation and by carrying out simulations with 10,000 replications.

Table 1. Exact and Simulated Characteristics of N_{m_0}

			Exact				Simulated			
ω	k	n_ω	Q_1	Me	Q_3	$E\{N_{m_0}\}$	Q_1	Me	Q_3	$E\{N_{m_0}\}$
0.5	5	25	22	30	40	31.81	21	30	39	31.51 ± 0.28
	10	25	22	27	34	29.04	22	27	33	28.31 ± 0.18
0.1	5	125	106	146	197	156.61	105	146	194	156.14 ± 1.38
	10	125	108	135	166	139.39	108	135	166	139.47 ± 0.87

3.3 The Expected Value and The LINEX Risk of $\tilde{\mu}_{N_{m_0}}$

Since $\bar{X}_n$ is independent of S_n^2, for all $n \geq 2$,

$$E_\theta\{\bar{X}_{N_{m_0}} \mid S_{m_0}^2\} = \mu \quad \text{w.p.1}, \tag{17}$$

and therefore, $E_\theta\{\bar{X}_{N_{m_0}}\} = \mu$, where $\theta = (\mu, \sigma^2)$. Substituting in (10) we obtain

$$E_\theta\{\tilde{\mu}_{N_{m_0}}\} = \mu - \frac{a}{2} E_{\sigma^2}\left\{\frac{S_{N_{m_0}}^2}{N_{m_0}}\right\}. \tag{18}$$

Thus, if $a > 0$ ($a < 0$) the bias of $\tilde{\mu}_{N_{m_0}}$ is negative (positive). Now, we derive a formula for $E_{\sigma^2}\left\{\frac{S_{N_{m_0}}^2}{N_{m_0}}\right\}$.

Lemma 1. *Under two-Stage sampling (9), we have*

$$\begin{aligned} E_{\sigma^2}\left\{\frac{S_{N_{m_0}}^2}{N_{m_0}}\right\} = \sigma^2 \Bigg[& \frac{1}{m_0} - 2k \sum_{j=0}^{\infty} \frac{1}{(m_0+j-1)(m_0+j)(m_0+j+1)} \\ & \times \left(P(k; \lambda_k(m_0+j)) + p(k; \lambda_k(m_0+j)\right) \\ & - \sum_{j=1}^{\infty} \frac{j}{(m_0+j-1)(m_0+j)(m_0+j+1)} P(k-1; \lambda_k(m_0+j)) \Bigg], \end{aligned} \tag{19}$$

where $\lambda_k = k/\lambda_{m_0}(\sigma^2)$ and $p(\cdot;\eta)$ is the probability mass function (p.m.f.) of a Poisson random variable with mean η.

For a proof, see Zacks and Mukhopadhyay (2009).

The LINEX risk of $\tilde{\mu}_{N_{m_0}}$ is

$$R(\tilde{\mu}_{N_{m_0}}) = E_\theta\left\{\exp\left(a\left(\bar{X}_{N_{m_0}} - \mu - \frac{aS_{N_{m_0}}^2}{2N_{m_0}}\right)\right)\right\} + \frac{a^2}{2} E_\theta\left\{\frac{S_{N_{m_0}}^2}{N_{m_0}}\right\} - 1. \tag{20}$$

The first term on the right-hand side of (20) is given in the following lemma.

Lemma 2.

$$E_\theta\left\{\exp\left(a(\bar{X}_{N_{m_0}}-\mu)-\frac{a^2S^2_{N_{m_0}}}{2N_{m_0}}\right)\right\}=\zeta_0\eta_0^{-\frac{1}{2}(m_0-1)}(1-P(k-1;\lambda_k m_0\eta_0))$$
$$+\sum_{j=1}^{\infty}\zeta_j\eta_j^{-\frac{1}{2}(m_0+j-1)}(P(k-1;\lambda_k(m_0+j-1)\eta_j)-P(k-1;\lambda_k(m_0+j)\eta_j)), \tag{21}$$

where

$$\eta_j=\left(1+\frac{a^2\sigma^2}{(m_0+j)(m_0+j-1)}\right),\quad j=0,1,\ldots \tag{22}$$

and

$$\zeta_j=\exp\left(\frac{a^2\sigma^2}{2(m_0+j)}\right),\quad j=0,1,\ldots \tag{23}$$

The risk $R(\tilde{\mu}_{N_{m_0}})$ is obtained from (19) and (20).

Notice that $\text{sgn}(\text{Bias}(\tilde{\mu}_{N_{m_0}}))=\text{sgn}(a)$ and the risk is independent of $\text{sgn}(a)$. In Table 2, we present some values of the bias and the LINEX risk of $\tilde{\mu}_{N_{m_0}}$.

Table 2: Exact Bias and LINEX Risk of $\tilde{\mu}_{N_{m_0}}$ under (9), $a=5$, $\sigma=1$

k	ω	n_ω	Bias($\tilde{\mu}_{N_{m_0}}$)	Risk($\tilde{\mu}_{N_{m_0}}$)
5	0.50	25	−0.07915	0.54119
	0.10	125	−0.01904	0.10172
	0.05	1250	−0.00980	0.05034
10	0.50	25	−0.08189	0.48938
	0.10	125	−0.01919	0.10137
	0.05	1250	−0.00980	0.05034
20	0.50	25	−0.06083	0.30703
	0.10	125	−0.01924	0.10128
	0.05	1250	−0.00981	0.05032

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