Multisource Bayesian sequential hypothesis testing

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1 Introduction and problem description

On some probability space $(\Omega, \mathbb{F}, \mathbb{P})$, let $(X^{(i)})_{t\ge 0}$, $1 \le i \le d$ be d independent Brownian motions with constant drifts $\mu^{(i)}$, $1 \le i \le d$, and $(T_n^{(j)}, Z_n^{(j)})_{n\ge 1}$, $1 \le j \le m$ be m independent compound Poisson processes independent of the Brownian motions $(X^{(i)})_{t\ge 0}$, $1 \le i \le d$. For every $1 \le j \le m$, $(T_n^{(j)})_{n\ge 1}$ are the arrival times, and $(Z_n^{(j)})_{n\ge 1}$ are the marks on some measurable space $(E, \mathcal{E})$, with arrival rates $\lambda^{(j)}$ and mark distributions $\nu^{(j)}(\cdot)$ on $(E, \mathcal{E})$.

Suppose that $\mu^{(i)}, 1 \le i \le d$ and $(\lambda^{(j)}, \nu^{(j)})_{1\le j\le m}$ are unknown, but exactly one of the following two simple hypotheses,

$$H_0 : \left\{ \begin{array}{c} \mu^{(i)} = \mu_0^{(i)}, 1 \le i \le d \\ (\lambda^{(j)}, \nu^{(j)}) = (\lambda_0^{(j)}, \nu_0^{(j)}), 1 \le j \le m \end{array} \right\}, \quad H_1 : \left\{ \begin{array}{c} \mu^{(i)} = \mu_1^{(i)}, 1 \le i \le d \\ (\lambda^{(j)}, \nu^{(j)}) = (\lambda_1^{(j)}, \nu_1^{(j)}), 1 \le j \le m \end{array} \right\},$$

is correct for some known $\mu_0^{(i)}$, $\mu_1^{(i)}$ for every $1 \le i \le d$, and $(\lambda_0^{(j)}, \nu_0^{(j)})$, $(\lambda_1^{(j)}, \nu_1^{(j)})$ for every $1 \le j \le m$, where probability measures $\nu_0^{(j)}$ and $\nu_1^{(j)}$ on $(E, \mathcal{E})$ are equivalent. Let Θ be the index of correct hypothesis, which is a $\{0,1\}$-valued random variable with prior distribution $\mathbb{P}\{\Theta = 1\} = 1 - \mathbb{P}\{\Theta = 0\} = \pi$ for some known $\pi \in (0,1)$.

The problem is to find a stopping time τ and a terminal decision rule d which depend only on the observations of Brownian motions $(X_n^{(i)})_{n\ge 0}$, $1 \le i \le d$ and compound Poisson processes $(T_n^{(j)}, Z_n^{(j)})_{n\ge 1}$, $1 \le j \le m$ and which have the smallest Bayes risk

$$R_{\tau,d}(\pi) := \mathbb{E}\left[\tau + 1_{\{\tau<\infty\}}\left(a 1_{\{d=0,\Theta=1\}} + \, b 1_{\{d=1,\Theta=0\}}\right)\right],$$

where a and b are known positive constants and correspond to the costs of making wrong terminal decisions. If such a decision rule (τ, d) exists, then it strikes optimal balance between the expected total sampling cost and the expected cost of selecting the wrong hypothesis. Similar problems arise when the efficacy of a new drug or a new medical procedure has to be determined in a clinical study, from which withdrawals of subjects at any time are always possible if their prognoses worsen.

The non-Bayes formulation of the sequential hypothesis testing problem has been studied by others and can be found in the recent reviews and contributions made by Lai [9,10], Dragalin, Tartakovsky, and Veeravalli [6,7], Lorden [11].

Bayesian sequential hypothesis testing problem was studied in discrete-time for the identification of the common distribution of i.i.d. observations by Wald and Wolfowitz [15], Blackwell and Girshick [1], Zacks [16], Shiryaev [14], in continuous-time for the identification of the drift of a Brownian motion by Shiryaev [14], for the identification of the arrival rate of a simple Poisson process by Peskir and Shiryaev [12,13], for the identification of the arrival rate and mark distribution of a compound Poisson process by Gapeev [8], Dayanik and Sezer [2], Dayanik, Poor, and Sezer [5]. The problem has not been addressed earlier for the joint identification of local characteristics of concurrently observed several independent Brownian motions and compound Poisson processes, and its solution is one of our contributions.

We show that an optimal decision rule (τ, d) always exists. The optimal stopping time τ is when the likelihood-ratio process

$$L_t := \exp\left\{\sum_{i=1}^{d}\left(\mu_1^{(i)}-\mu_0^{(i)}\right)\left(X_t^{(i)}-X_0^{(i)}\right)-\frac{t}{2}\sum_{i=1}^{d}\left[\left(\mu_1^{(i)}\right)^2-\left(\mu_0^{(i)}\right)^2\right]\right\}$$
$$\times \exp\left\{\sum_{j=1}^{m}\sum_{n:0<T_n^{(j)}\le t}\log\left(\frac{\lambda_1^{(j)}}{\lambda_0^{(j)}}\frac{\mathrm{d}\nu_1^{(j)}}{\mathrm{d}\nu_0^{(j)}}\left(Z_n^{(j)}\right)\right)-t\sum_{j=1}^{m}\left(\lambda_1^{(j)}-\lambda_0^{(j)}\right)\right\}$$

exits for the first time a bounded interval $(\phi_1(1-\pi)/\pi, \phi_2(1-\pi)/\pi)$ for some suitable constants $0<\phi_1<b/a<\phi_2<\infty$, and optimal terminal decision rule d is to choose the null hypothesis if $\pi L_\tau/(1-\pi)\le b/a$ and the alternative hypothesis otherwise. We describe a provably convergent numerical method to calculate both the minimum Bayes risk and the decision boundaries ϕ_1 and ϕ_2 of the optimal stopping rule τ. The minimum Bayes risk is shown to be the uniform limit of a decreasing sequence of successive approximations, which are obtained by applying a contraction mapping iteratively to a suitable initial function. The maximum absolute difference between successive approximations is bounded by an explicit bound, which decays at a known exponential rate with the number of iterations. Thus, one can always determine the necessary number of iterations ex-ante for any desired level of accuracy in the approximations of the minimum Bayes risk and optimal decision boundaries.

We address the problem by reducing it to the optimal stopping of the jump-diffusion likelihood-ratio process. The method strips the jumps away from the diffusion part and applies the potential-theoretic direct solution method developed by Dayanik and Karatzas [4] and Dayanik [3]. The solution method developed here can also be applied effectively to price American-type financial contracts and real options with jump-diffusion underlyers.

2 A model

Let $(\Omega, \mathcal{F}, \mathbb{P}_0)$ be a probability space hosting the following independent stochastic elements: (i) X is a Brownian motion with drift rate μ_0, (ii) $(T_n, Z_n)_{n\ge1}$ is a compound Poisson process with arrival rate λ_0 and mark distribution ν_0 on $(E, \mathcal{E})$, and (iii) Θ is a Bernoulli r.v. with success probability $\pi\in(0,1)$.

We denote by $\mathbb{G}=(\mathcal{G}_t)_{t\ge0}$ the filtration obtained by enlarging the observation filtration $\mathbb{F}$ with the information about Θ; i.e., $\mathcal{G}_t := \mathcal{F}_t\vee\sigma(\Theta)$ for every $t\ge0$, and introduce the likelihood ratio process

$$L_t = \exp\left\{(\mu_1-\mu_0)(X_t-X_0)-\left[\frac{\mu_1^2-\mu_0^2}{2}+\lambda_1-\lambda_0\right]t+\sum_{0<T_n\le t}\log\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(Z_n)\right)\right\}, t\ge0.$$

Let $\mathbb{P}$ be a new probability measure on $(\Omega, \mathcal{G}_\infty)$, whose restriction to each $\mathcal{G}_t$, $t\ge0$ is defined in terms of the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_0}\right|_{\mathcal{G}_t} = \xi_t := 1_{\{\Theta=0\}}+1_{\{\Theta=1\}}L_t, \quad t\ge0. \tag{1}$$

Under probability measure $\mathbb{P}$ defined by (1), we have the same setup as in the problem description. Therefore, in the remainder we will work with the model constructed here.

Starting from any arbitrary but fixed initial state $\phi\in\mathbb{R}_+$, let us define the process

$$\Phi_0=\phi \qquad \text{and} \qquad \Phi_t=\Phi_0 L_t, \qquad t\ge0. \tag{2}$$

The Bayes theorem implies that

$$\frac{\mathbb{P}\{\Theta=1\mid\mathcal{F}_t\}}{\mathbb{P}\{\Theta=0\mid\mathcal{F}_t\}}=\frac{\mathbb{E}_0\left[\xi_t 1_{\{\Theta=1\}}\mid\mathcal{F}_t\right]}{\mathbb{E}_0\left[\xi_t 1_{\{\Theta=0\}}\mid\mathcal{F}_t\right]}=\frac{L_t\mathbb{P}_0\{\Theta=1\mid\mathcal{F}_t\}}{\mathbb{P}_0\{\Theta=0\mid\mathcal{F}_t\}}=\frac{\pi}{1-\pi}L_t,\ t\ge0 \quad \mathbb{P}^{\frac{\pi}{1-\pi}}\text{-a.s.}$$

Proposition 1. *The Bayes risk $R_{\tau,d}(\pi)$ can be written as*

$$R_{\tau,d}(\pi) = b(1-\pi)\mathbb{P}_0^{\frac{\pi}{1-\pi}}\{\tau<\infty\} + (1-\pi)\mathbb{E}_0^{\frac{\pi}{1-\pi}}\left[\int_0^\tau (1+\Phi_t)\mathrm{d}t + (a\Phi_\tau - b)\,1_{\{d=0,\tau<\infty\}}\right], \quad (3)$$

where $\mathbb{P}_0^\phi$ is the probability $\mathbb{P}_0$ with $\Phi_0=\phi$, and $\mathbb{E}_0^\phi$ is the expectation with respect to $\mathbb{P}_0^\phi$. If we define $d(t) := 1_{(b/a,\infty)}(\Phi_t)$, $t \geq 0$, then the pair $(\tau, d(\tau))$ belongs to Δ. We have $R_{\tau,d}(\pi) \geq R_{\tau,d(\tau)}(\pi)$ for every $(\tau,d)\in\Delta$ and $\pi\in(0,1)$, and the minimum Bayes risk $U(\pi)$ an be written as $U(\pi) \equiv \inf_{(\tau,d)\in\Delta} R_{\tau,d}(\pi) = b(1-\pi) + (1-\pi)V(\frac{\pi}{1-\pi})$, $\pi\in(0,1)$ in terms of the value function $V(\cdot)$ of the auxiliary optimal stopping problem

$$V(\phi) := \inf_{\tau\in\mathbb{F}} \mathbb{E}_0^\phi\left[\int_0^\tau g(\Phi_t)\mathrm{d}t + 1_{\{\tau<\infty\}}h(\Phi_\tau)\right], \qquad \phi\geq 0, \quad (4)$$

where the running cost function $g:\mathbb{R}_+\mapsto\mathbb{R}$ and the terminal cost function $h:\mathbb{R}_+\mapsto\mathbb{R}$ are defined by $g(\phi):=1+\phi$ and $h(\phi):=-(a\phi-b)^-$.

Let us denote the point process generated by $(T_n,Z_n)_{n\geq1}$ with $p((0,t]\times B)=\sum_{n=1}^\infty 1_{(0,t]\times B}(T_n,Z_n)$, $t\geq 0$. An application of Itô's rule gives the dynamics of process L as

$$L_0 = 1, \quad \text{and} \quad \mathrm{d}L_t = (\mu_1-\mu_0)L_t(\mathrm{d}X_t - \mu_0\mathrm{d}t) + L_{t-}\int_E\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)-1\right)[p(\mathrm{d}t\times\mathrm{d}z) - \nu_0(\mathrm{d}z)\lambda_0\mathrm{d}t], \quad t\geq 0.$$

Because of (2), the dynamics of process Φ becomes

$$\Phi_0 = \phi, \quad \text{and} \quad \mathrm{d}\Phi_t = (\mu_1-\mu_0)\Phi_t(\mathrm{d}X_t - \mu_0\mathrm{d}t) + \Phi_{t-}\int_E\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)-1\right)[p(\mathrm{d}t\times\mathrm{d}z) - \nu_0(\mathrm{d}z)\lambda_0\mathrm{d}t], \quad t\geq 0,$$

If we define

$$Y_t^{k,\phi} := \phi\exp\left\{(\mu_1-\mu_0)(X_t^{(k)} - X_0^{(k)}) - \left[\frac{\mu_1^2-\mu_0^2}{2} + \lambda_1 - \lambda_0\right]t\right\}, \; t\geq 0, \; k\geq 0, \; \phi\geq 0,$$

then the sample paths of the conditional odds-ratio process Φ can be decomposed into diffusion and jump parts as in

$$\Phi_t = \begin{cases} Y_{t-T_k}^{k,\Phi_{T_k}}, & \text{if } t\in[T_k,T_{k+1}) \text{ for some } k\geq 0, \\ \dfrac{\lambda_1}{\lambda_0}\dfrac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(Z_{k+1})Y_{T_{k+1}-T_k}^{k,\Phi_{T_k}}, & \text{if } t = T_{k+1} \text{ for some } k\geq 0. \end{cases} \quad (5)$$

The process $Y^{k,\phi}$ is a diffusion with dynamics

$$Y_0^{k,\phi} = \phi, \quad \text{and} \quad \mathrm{d}Y_t^{k,\phi} = (\mu_1-\mu_0)Y_t^{k,\phi}(\mathrm{d}X_t^{(k)} - \mu_0\mathrm{d}t) + (\lambda_0-\lambda_1)Y_t^{k,\phi}\mathrm{d}t, \quad t\geq 0.$$

3 Jump operator and successive approximations

We will denote Y^{0,Φ_0} by Y^{Φ_0}, which is a diffusion with dynamics

$$Y_0^{\Phi_0} = \Phi_0, \qquad \mathrm{d}Y_t^{\Phi_0} = (\lambda_0-\lambda_1)\mathrm{d}t + (\mu_1-\mu_0)Y_t^{\Phi_0}(\mathrm{d}X_t - \mu_0\mathrm{d}t) \quad t\geq 0. \quad (6)$$

and $(\mathbb{P}_0,\mathbb{F})$-infinitesimal generator $(A_0w)(\phi) = (\lambda_0-\lambda_1)\phi w'(\phi) + \frac{1}{2}(\mu_1-\mu_0)^2\phi^2 w''(\phi)$ acting on twice-continuously differentiable functions $w:\mathbb{R}_+\mapsto\mathbb{R}$. For every bounded function $w:\mathbb{R}_+\mapsto\mathbb{R}$, let

$$(Kw)(\phi) := \int_E w\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\phi\right)\nu_0(\mathrm{d}z), \quad \phi\in\mathbb{R}_+, \quad (7)$$

and the jump operator

$$(Jw)(\phi) := \inf_{\tau\in\mathbb{F}^X} \mathbb{E}_0^\phi \left[\int_0^\tau e^{-\lambda_0 t} \left[g(Y_t^{\Phi_0}) + \lambda_0 (Kw)(Y_t^{\Phi_0}) \right] \mathrm{d}t + e^{-\lambda_0 \tau} h(Y_\tau^{\Phi_0}) \right], \quad \phi \in \mathbb{R}_+, \qquad (8)$$

which is itself a discounted optimal stopping problem for the diffusion Y^{Φ_0} in (6), with discount rate λ_0, running cost function $g(\cdot) + \lambda_0(Kw)(\cdot)$ and terminal cost function $h(\cdot)$.

Proposition 2. *Let us define $v_0(\cdot) := h(\cdot)$ and $v_n(\cdot) := (Jv_{n-1})(\cdot)$, $n \geq 1$. The sequence $(v_n(\cdot))_{n\geq 0}$ is decreasing, and the pointwise limit $v_\infty(\cdot) := \lim_{n\to\infty} v_n(\cdot) \equiv \inf_{n\geq 0} v_n(\cdot)$ exists. Every $v_n(\cdot)$, $n \geq 0$ and $v_\infty(\cdot)$ are nondecreasing, concave, and bounded between $-b$ and $h(\cdot)$.*

Proposition 3. *The function $v_\infty(\cdot)$ is the largest solution of the equation $v(\cdot) = (Jv)(\cdot)$ less than or equal to $h(\cdot)$.*

Theorem 1. *The value function $V(\cdot)$ and the limit $v_\infty(\cdot)$ of successive approximations are the same.*

4 Numerical algorithm and examples

In Figure 1, we describe a numerical algorithm to calculate the successive approximations $v_n(\cdot)$, $n \geq 0$ of the value function $V(\cdot) \equiv v_\infty(\cdot)$ of the auxiliary optimal stopping problem in (4) and Bayes ε-optimal decision rules for the Bayesian sequential binary hypothesis testing problem. In the examples described below and illustrated in Figure 2, that algorithm is used to calculate the approximations $v_n(\cdot)$, $n \geq 0$ until the maximum absolute difference between successive approximations is reduced below an acceptable level.

Nine panels in Figure 2 display the approximate value functions and minimum Bayes risks corresponding to nine examples. In each example, the observation process consists of a Brownian motion X with drift μ and a simple Poisson process $(T_n)_{n\geq 1}$ (i.e., marks Z_n, $n \geq 1$ are known and equal to one almost surely) with arrival rate λ. Under the null hypothesis H_0, we assume that the unknown drift and arrival rates are equal to $\mu_0 = 0$ and $\lambda_0 = 1$, respectively. We also assume that the costs of wrongly choosing H_0 and H_1 are the same and equal to $a = b = 0.5$. However, drift rate μ_1 and arrival rate λ_1 under alternative hypothesis H_1 are different in nine examples; drift rate μ_1 takes values 2, 3, 4 along three columns, respectively, and arrival rate λ_1 takes values $7, 9, 11$ along three rows, respectively. Each panel is divided in two parts. The upper part shows the optimal Bayes risk $U(\cdot)$ on $[0, 1]$ displayed on the upper horizontal axis, and the lower part shows the value function $V(\cdot)$ of the auxiliary optimal stopping problem in (4) on $\mathbb{R}_+$ displayed on the lower horizontal axis. Both $U(\cdot)$ and $V(\cdot)$ are plotted with solid curves. These functions are compared with $U_p(\cdot)$, $V_p(\cdot)$, $U_X(\cdot)$, and $V_X(\cdot)$, where $U_p(\cdot)$ and $U_X(\cdot)$ are obtained by taking the infimum over the stopping times of smaller natural filtrations $\mathbb{F}^p$ and $\mathbb{F}^X$ of Poisson process and Brownian motion, respectively. On the other hand, $V_p(\cdot)$ and $V_X(\cdot)$ are the value functions of the optimal stopping problems analogous to (4); i.e.,

$$V_p(\phi) := \inf_{\tau\in\mathbb{F}^p} \mathbb{E}_0^\phi \left[\int_0^\tau g(\Phi_t^{(p)}) \mathrm{d}t + 1_{\{\tau<\infty\}} h(\Phi_\tau^{(p)}) \right], \qquad \phi \geq 0,$$

$$V_X(\phi) := \inf_{\tau\in\mathbb{F}^X} \mathbb{E}_0^\phi \left[\int_0^\tau g(\Phi_t^{(X)}) \mathrm{d}t + 1_{\{\tau<\infty\}} h(\Phi_\tau^{(X)}) \right], \qquad \phi \geq 0,$$

where

$$\Phi_t^{(p)} := \frac{\mathbb{P}\{\Theta \leq t \mid \mathcal{F}_t^p\}}{\mathbb{P}\{\Theta > t \mid \mathcal{F}_t^p\}} \qquad \text{and} \qquad \Phi_t^{(X)} := \frac{\mathbb{P}\{\Theta \leq t \mid \mathcal{F}_t^X\}}{\mathbb{P}\{\Theta > t \mid \mathcal{F}_t^X\}} \qquad \text{for every } t \geq 0.$$

The functions $U_p(\cdot)$, $V_p(\cdot)$ and $U_X(\cdot)$, $V_X(\cdot)$ are related each other in the same way as $U(\cdot)$, $V(\cdot)$ are related. The differences in the Bayes risks $U_p(\cdot)$, $U_X(\cdot)$, and $U(\cdot)$ are due to the contributions of observing the processes X and $(T_n)_{n\geq 1}$ separately or simultaneously to the efforts to identify the correct

Initialization Set $v_0(\phi) = h(\phi)$ and $\underline{w}(\phi) = -b$ for every $\phi > 0$. Calculate $F(\phi) = \psi(\phi)/\eta(\phi) = \phi^{\alpha_1-\alpha_0}$ for every $\phi > 0$. Set $n = 1$.

Step 1 Calculate

$$(Lv_n)(\zeta) = \left(\frac{Hv_n - h}{\eta}\right) \circ F^{-1}(\zeta), \quad \zeta \geq 0.$$

Step 2 Calculate critical boundaries $\zeta_1[v_n]$ and $\zeta_2[v_n]$, which are unique solutions of

$$(Lv_n)'(\zeta_1[v_n]) = \frac{(Lv_n)(\zeta_2[v_n]) - (Lv_n)(\zeta_1[v_n])}{\zeta_2[v_n] - \zeta_1[v_n]} = (Lv_n)'(\zeta_2[v_n])$$

Recall that $0 < \zeta_1[\underline{w}] \leq \zeta[v_n] \leq F(b/a) \leq \zeta_2[v_n] \leq \zeta_2[\underline{w}] < \infty$, and the lower bound $\zeta_1[\underline{w}]$ and upper bound $\zeta_2[\underline{w}]$ on the critical boundaries $\zeta_1[v_n]$ and $\zeta_2[v_n]$ for $n \in \{1, 2, \ldots\} \cup \{\infty\}$ are useful to control the computer memory. Calculate the smallest nonnegative concave majorant $(Mv_n)(\cdot)$ of $(Lv_n)(\cdot)$ on $\mathbb{R}_+$ by

$$(Mv_n)(\zeta) = \begin{cases} (Lv_n)(\zeta), & \text{if } \zeta \in [0, \zeta_1[v_n]] \cup [\zeta_2[v_n], \infty), \\ \dfrac{\zeta_2[v_n] - \zeta}{\zeta_2[v_n] - \zeta_1[v_n]}(Lv_n)(\zeta_1[v_n]) \\ \quad + \dfrac{\zeta - \zeta_1[v_n]}{\zeta_2[v_n] - \zeta_1[v_n]}(Lv_n)(\zeta_2[v_n]), & \text{if } \zeta \in (\zeta_1[v_n], \zeta_2[v_n]). \end{cases}$$

Step 3 Calculate $\phi_1[v_n] = F^{-1}(\zeta_1[v_n])$ and $\phi_2[v_n] = F^{-1}(\zeta_2[v_n])$ and

$$(Gv_n)(\phi) = \begin{cases} (Hv_n)(\phi) - h(\phi), \quad \phi \in (0, \phi_1[v_n]] \cup [\phi_2[v_n], \infty), \\ \dfrac{(\phi_2[v_n])^{\alpha_1-\alpha_0} - \phi^{\alpha_1-\alpha_0}}{(\phi_2[v_n])^{\alpha_1-\alpha_0} - (\phi_1[v_n])^{\alpha_1-\alpha_0}}(Hv_n - h)(\phi_1[v_n]) \\ \quad + \dfrac{\phi^{\alpha_1-\alpha_0} - (\phi_1[v_n])^{\alpha_1-\alpha_0}}{(\phi_2[v_n])^{\alpha_1-\alpha_0} - (\phi_1[v_n])^{\alpha_1-\alpha_0}}(Hv_n - h)(\phi_2[v_n]), & \phi \in (\phi_1[v_n], \phi_2[v_n]). \end{cases}$$

Step 4 Calculate $v_{n+1}(\phi) = (Jv_n)(\phi) = (Hv_n)(\phi) - (Gv_n)(\phi)$ for every $\phi > 0$.

Step 5 If some stopping criterion has not yet been satisfied (e.g., the uniform bound $b\beta^n$ on $\|v_\infty - v_n\|$ has not yet been reduced below some desired error level), then set n to $n + 1$ and go to **Step 1**, otherwise stop.

Outcome After the algorithm terminates with $v_{n+1}(\cdot)$, $\phi_1[v_n]$, and $\phi_2[v_n]$,

1. we have $v_{n+1}(\phi) - b\beta^{n+1} \leq V(\phi) \leq v_{n+1}(\phi)$ for every $\phi > 0$,
2. the stopping time $\widetilde{\tau}[v_n] = \inf\{t \geq 0;\ \Phi_t \notin (\phi_1[v_n], \phi_2[v_n])\}$ is ε-optimal for every $\varepsilon > b\beta^{n+1}$ for the auxiliary optimal stopping problem in (4); i.e.,

$$V(\phi) \leq \mathbb{E}_0^\phi\left[\int_0^{\widetilde{\tau}[v_n]}(1 + \Phi_t)\mathrm{d}t + h(\Phi_{\widetilde{\tau}[v_n]})\right] \leq V(\phi) + b\beta^{n+1},\ \phi > 0,$$

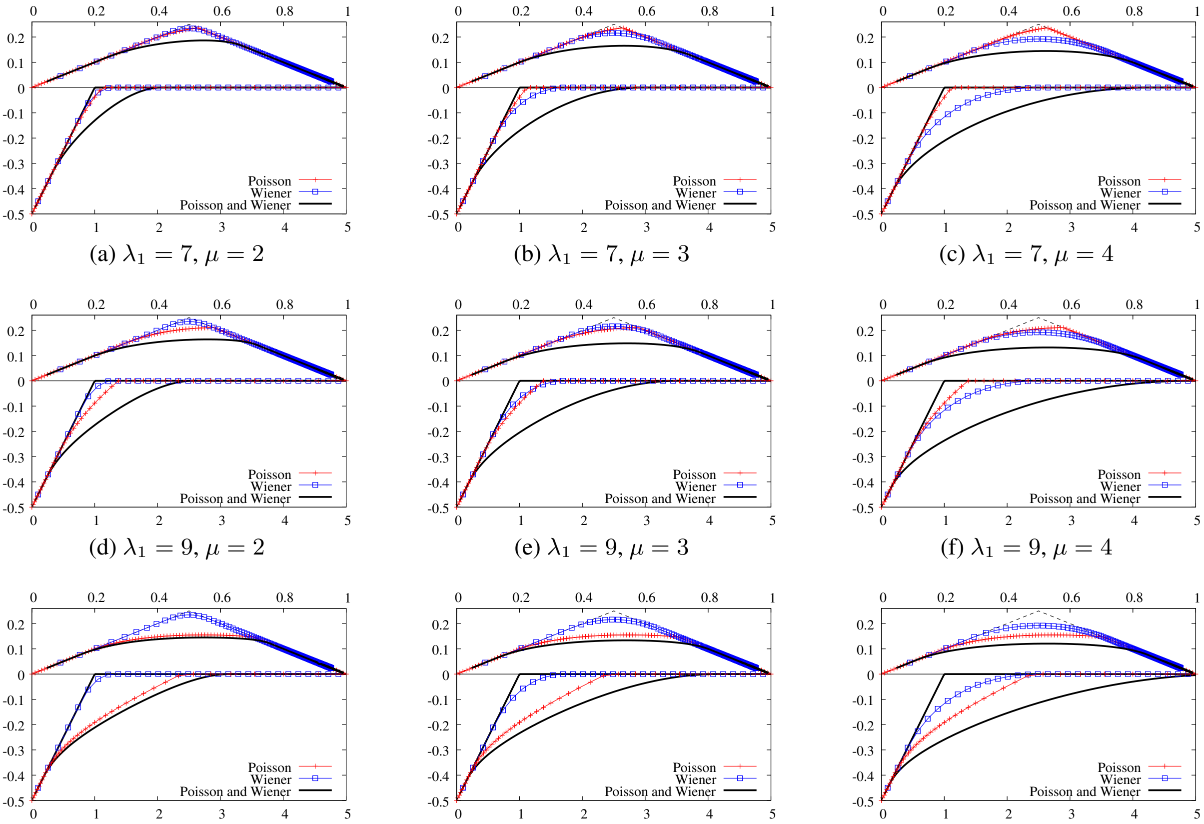
3. the decision rule $(\widetilde{\tau}[v_n], d(\widetilde{\tau}[v_n]))$ is Bayes ε-optimal for every $\varepsilon > b\beta^{n+1}$ for the Bayesian sequential binary hypothesis testing problem ; i.e.,

$$U(\pi) \leq R_{\widetilde{\tau}[v_n], d(\widetilde{\tau}[v_n])}(\pi) \leq U(\pi) + b\beta^{n+1},\ \pi \in (0, 1).$$

Fig. 1. Numerical algorithm to solve the Bayesian sequential binary hypothesis testing problem.

hypothesis about the drift rate X and arrival rate of $(T_n)_{n\geq 1}$. Sometimes, Poisson process observations provide more information than Brownian motion observations as in (d), (e), (g), (h), and (i). Sometimes, Brownian motion observations provide more information than Poisson process observations as in (a), (b), and (f). In every case, however, observing both Poisson process and Brownian motion at the same time provides more information, which is often significantly more than two processes provide separately, as in (a), (b), (c), (d), (e), (f), and (i).

Intuitively, we expect the contributions of both information sources, observed separately or concurrently, to increase as μ_1 and λ_1 get farther away from $\mu_0 = 0$ and $\lambda_0 = 1$, respectively, and the results reported in Figure 2 support these expectations. Indeed, the Bayes risks $U_p(\cdot)$ and $U(\cdot)$ shift downward across along the rows, and $U_X(\cdot)$ and $U(\cdot)$ do the same along the columns.



(a) $\lambda_1 = 7$, $\mu = 2$ (b) $\lambda_1 = 7$, $\mu = 3$ (c) $\lambda_1 = 7$, $\mu = 4$

(d) $\lambda_1 = 9$, $\mu = 2$ (e) $\lambda_1 = 9$, $\mu = 3$ (f) $\lambda_1 = 9$, $\mu = 4$

(g) $\lambda_1 = 11$, $\mu = 2$ (h) $\lambda_1 = 11$, $\mu = 3$ (i) $\lambda_1 = 11$, $\mu = 4$

Fig. 2. In all of nine examples above, $\lambda_0 = 1$, $\mu_0 = 0$, and the cost parameters are $a = b = 0.5$. We also assume that jumps are of unit size under both hypotheses.

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