The Spend-It-All Region and Small Time Results for the Continuous Bomber Problem

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Abstract. A problem of optimally allocating partially effective ammunition x to be used on randomly arriving enemies in order to maximize an aircraft's probability of surviving for time t, known as the Bomber Problem, was first posed by Klinger and Brown (1968). They conjectured a set of apparently obvious monotonicity properties of the optimal allocation function $K(x, t)$. Although some of these conjectures, and versions thereof, have been proved or disproved by other authors since then, the remaining central question, that $K(x, t)$ is nondecreasing in x, remains unsettled. After reviewing the problem and summarizing the state of these conjectures, in the setting where x is continuous we discuss a proof the existence of a "spend-it-all" region in which $K(x, t) = x$ and find its boundary, inside of which the long-standing, unproven conjecture of monotonicity of $K(\cdot, t)$ holds. A new approach is then taken of directly estimating $K(x, t)$ for small t, providing a complete small-t asymptotic description of $K(x, t)$ and the optimal probability of survival.

Keywords. Ammunition rationing, Poisson process, sequential optimization.

1 Introduction

Klinger and Brown (1968) introduced a problem of optimally allocating partially effective ammunition to be used on enemies arriving at a Poisson rate in order to maximize the probability that an aircraft (hereafter "the bomber") survives for time t, known as the Bomber Problem. Given an amount x of ammunition, let $K(x, t)$ denote the optimal amount of ammunition the bomber would use upon confronting an enemy at *time* t, defined as the time remaining to survive. The appearance of enemies is driven by a time-homogeneous Poisson process of known rate, taken to be 1. An enemy survives the bomber's expenditure of an amount $y \in [0, x]$ of its ammunition with the geometric probability q^y, for some known $q \in (0, 1)$, after which the enemy has a chance to destroy the bomber, which happens with known probability $v \in (0, 1]$ (the $v = 0$ case being trivial). By rescaling x, we assume without loss of generality that $q = e^{-1}$, and hence the probability that the bomber survives an enemy encounter in which it spends an amount y of its ammunition is

$$a(y) = 1 - ve^{-y}. \tag{1}$$

Klinger and Brown (1968) posed two seemingly obvious conjectures about the optimal allocation function $K(x, t)$:

A: $K(x, t)$ is nonincreasing in t for all fixed $x \geq 0$;
B: $K(x, t)$ is nondecreasing in x for all fixed $t \geq 0$.

Klinger and Brown (1968) showed that [B] implies [A] when $v = 1$, although, as will be discussed below, [B] remains in doubt. Improving the situation, Samuel (1970) showed that [A] holds without assuming [B] in the setting where units of ammunition x are discrete, and in this setting also showed that a third conjecture holds:

C: The amount $x - K(x, t)$ held back by the bomber is nondecreasing in x for all fixed $t \geq 0$.

[C] was first stated as a formal property by Simons and Yao (1990), who claimed that it can be shown to hold for continuous x and t by arguments similar to the ones they provide for a case where both

x and t are discrete, and they also make theoretical and computational progress toward [B] in various discrete/continuous settings. Also in the setting where both x and t are continuous, Bartroff, Goldstein, Rinott, and Samuel-Cahn (2009) recently showed that [A] holds, and provide a full proof of [C] in this setting. Weber (1985) considered an infinite-horizon variant of the Bomber Problem in which the objective is to maximize the number of enemies shot down (thus removing t from the problem) and found that, for discrete x, the property related to [B], that of monotonicity of $K(x)$, fails to hold. Shepp et al. (1991) considered the infinite-horizon problem for continuous x and reached the same conclusion. On the other hand, Bartroff et al. (2009) consider the variation of the problem where the bomber is invincible, and both x and t are present and continuous, and show that [B] holds.

In spite of the results of Weber (1985), Shepp et al. (1991), and Bartroff et al. (2009), conjecture [B] has not been settled in any close relative to the original Bomber Problem, and it remains the conjecture about which the least is known. To gain insight into the function $K(x,t)$, perhaps as a step towards resolving [B] in greater generality, we discuss here a new approach to the Bomber Problem of directly estimating, or when possible solving for, $K(x,t)$ when both x and t are continuous. One might expect *a priori* that if x or t is sufficiently small then the optimal strategy is to spend all or nearly all of the available ammunition x, i.e., that $K(x,t)$ is equal to or nearly x. On the other hand, since the ammunition is assumed to be continuous it is not obvious that there exists a "spend-it-all" region where $K(x,t)$ is *identically* x. In Section 2 we discuss a result showing that there is indeed a spend-it-all region of (x,t) values for which $K(x,t) = x$ and where [B] holds, and we give an estimate of the region's boundary in Theorem 1, and are able to find it exactly in most cases. However, in Section 3 we discuss a result that shows that there are many other regimes in which $K(x,t)$ is not so simple, but can nevertheless be described asymptotically for small values of t. In particular, in Theorem 2 we give a characterization of the asymptotic behavior of $K(x,t)$ for small t which shows that regardless of how small t is, there are large intervals of x values for which $K(x,t)/x$ approaches any, even arbitrarily small, positive fraction, in stark contrast to the spend-it-all strategy. The relation of these results to the outstanding conjecture [B] and extensions are discussed in Section 4.

2 The Spend-it-all region

In this section we describe an (x,t)-region where $K(x,t)$ is identically x, the so-called "spend-it-all" region. The boundary of this region can be solved for, exactly as (2), except for a special configuration of the parameters x, t, v in which the boundary can be estimated from both sides; see (9).

In what follows, let $u = 1 - v \in [0,1)$ denote the probability that the bomber survives an enemy's counterattack, let $P(x,t)$ denote the optimal probability of survival at time t when the bomber has ammunition x, and let $H(x,t)$ denote the optimal conditional probability of survival given an enemy at time t, with ammunition x.

Theorem 1. *For $u \in (0,1)$ and $t > 0$ define*

$$g_u(t) = \log[1 + u/(e^{tu} - 1)], \tag{2}$$

and extend this definition to $u = 0$ by defining

$$g_0(t) = \lim_{u\to 0} g_u(t) = \log(1 + t^{-1}).$$

For $u \in [0,1)$ and $t > 0$ define

$$f_u(t) = \log(1 + t^{-1} - u). \tag{3}$$

If $u \in [0,1)$ and $t > 0$ satisfy one of the following:

$$(i)\;\; u = 0, \tag{4}$$

$$(ii)\;\; u \in (0,1/2) \quad \textit{and} \quad t \geq u^{-1}\log(2v), \tag{5}$$

$$(iii)\;\; u \in [1/2,1), \tag{6}$$

then

$$K(x,t) = x \text{ if and only if } x \le g_u(t). \tag{7}$$

In the remaining case, where

$$u \in (0, 1/2) \quad \text{and} \quad t < u^{-1}\log(2v), \tag{8}$$

we have

$$K(x,t) = x \text{ if } x \le f_u(t), \text{ and } K(x,t) < x \text{ if } x > g_u(t). \tag{9}$$

The theorem, proved elsewhere, may be summarized by saying that, except for the configuration of t, u values in (8), the spend-it-all region's boundary is given exactly by $g_u(t)$, which is positive for all $t > 0$ and approaches 0 as $t \to \infty$. Although the authors conjecture that $g_u(t)$ is the boundary of the spend-it-all region for all $t > 0$ and $u \in [0, 1)$, this has not been shown in the remaining case (8). Instead, in this case the boundary can be estimated from above by $g_u(t)$ and from below by $f_u(t)$, which is strictly less than $g_u(t)$ for all $t > 0$ but asymptotically equivalent to it as $t \to 0$. Although $f_u(t)$ is negative for $t > u^{-1}$, it is utilized as a bound only when (8) holds, in which case $u^{-1} > u^{-1}\log(2v) > 0$; see Figure 1. A consequence of the theorem is that, regardless of the value of u, for any $x > 0$ there is t sufficiently small such that the optimal strategy spends it all (i.e., $K(x,t) = x$), and for any $t > 0$ there is x sufficiently small such that the optimal strategy spends it all.

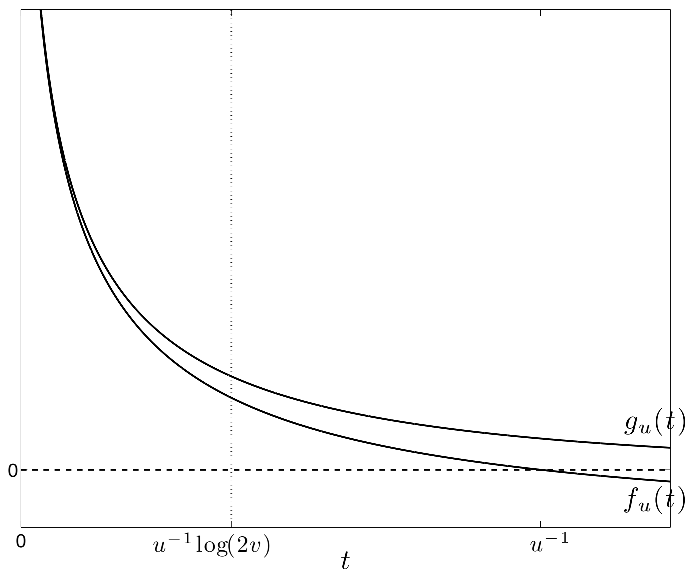

Figure 1: The bounds $f_u(t)$ and $g_u(t)$ on the spend-it-all region, and the switch-over point $u^{-1}\log(2v)$. Pictured is the $u \in (0, 1/2)$ case, in which $0 < u^{-1}\log(2v) < u^{-1}$.

3 An Asymptotic characterization of $K(x,t)$

In this section we discuss an asymptotic description of the optimal allocation function $K(x,t)$ as $t \to 0$, and for this purpose it suffices to consider sequences (x,t) with $t \to 0$. In addition to giving an asymptotic description of the optimal survival probability $P(x,t)$ and the optimal conditional survival probability $H(x,t)$, our main goal is to characterize the fraction $K(x,t)/x$ of the current ammunition x spent by the optimal strategy at time t, and it turns out that $K(x,t)/x$ approaches a finite nonzero limit on sequences (x, t) such that $|\log t| / x$ approaches a finite nonzero limit. We thus give an essentially complete asymptotic description of $K(x, t)$ by considering sequences $(x, t) = (x_t, t)$ such that

$$\frac{|\log t|}{x} \to \rho \in (0, \infty) \quad \text{as } t \to 0, \tag{10}$$

leaving divergent sequences to be handled by considering subsequences. Note that a consequence of (10) is that $x \to \infty$ at the same rate at which $|\log t| \to \infty$ as $t \to 0$. It should perhaps not be surprising that this is the nontrivial asymptotic regime since the boundary of the spend-it-all region given in Theorem 1 is asymptotically equivalent to $|\log t|$ as $t \to 0$. To streamline the notation in the statement of the theorem, let $\binom{1}{2}^{-1} = \infty$.

Theorem 2. *Under (10), let $j \in \{1, 2, \dots\}$ be such that*

$$\binom{j+1}{2}^{-1} \le \rho < \binom{j}{2}^{-1}. \tag{11}$$

Then, as $t \to 0$,

$$\frac{K(x,t)}{x} \to 1/j + \rho(j-1)/2 \tag{12}$$

$$\frac{1}{x} |\log(1 - H(x,t))| \to 1/j + \rho(j-1)/2 \tag{13}$$

$$\frac{1}{x} |\log(1 - P(x,t))| \to 1/j + \rho(j+1)/2. \tag{14}$$

We briefly discuss this result. Note that the j satisfying (11) is nonincreasing in ρ and, in particular, $\rho \ge 1$ corresponds to $j = 1$ while $\rho < 1$ corresponds to $j > 1$. The right hand sides of (12) and (13) equal 1 for $j = 1$, and are in the interval $[2/(j+1), 2/j)$ for $j \ge 2$; similarly, the right hand side of (14) is in the interval $[2/j, 2/(j-1))$ for all $j \ge 1$. In particular, (12) implies that $K(x,t)/x$ can take on any value in $(0, 1]$. The rates of convergence in (12)-(14) are functions of the rate of convergence in (10). Specifically, without assuming more than $|\log t| - \rho x = o(x)$ in (10), the same $o(x)$ term appears in the convergence of $K(x,t)$, $|\log(1 - H(x,t))|$, and $|\log(1 - P(x,t))|$ in (12)-(14). However, when $\rho > 1$, the convergence is $O(1/x)$ in (12) and (13), but in no other cases, an artifact of the natural upper bound $K(x,t) \le x$ that is relevant only in the $\rho > 1$ case.

The result (12) can equivalently be stated as, under (10),

$$K(x,t) \sim \frac{x}{j} + \left(\frac{j-1}{2}\right) |\log t| \tag{15}$$

as $t \to 0$ for j satisfying (11). Hence, for small t, the first quadrant of the (x,t)-plane can be thought of as partitioned into the regions

$$R_j = \left\{ (x,t) : \quad x > 0, \quad t > 0, \quad \binom{j+1}{2}^{-1} \le \frac{|\log t|}{x} < \binom{j}{2}^{-1} \right\}, \quad j = 1, 2, \dots, \tag{16}$$

which determine the asymptotic behavior of the optimal strategy. Figure 2 plots (15) and the boundaries of the first few R_j. Note that although (15) varies smoothly within each R_j, it is continuous but not smooth at the lower boundary of R_j. For small t, $K(x,t)$ given by (12) turns out to be such that if $(x,t) \in R_j$, then after firing $K(x,t)$ at an immediate enemy, the new state $(x - K(x,t), t)$ lies in R_{j-1}. This leads to the inductive method of proof, given elsewhere. The boundary of the R_1 region is asymptotically equivalent to the estimates of the spend-it-all region's boundary in Theorem 1 in the strong sense that their difference is $o(1)$ as $t \to 0$.

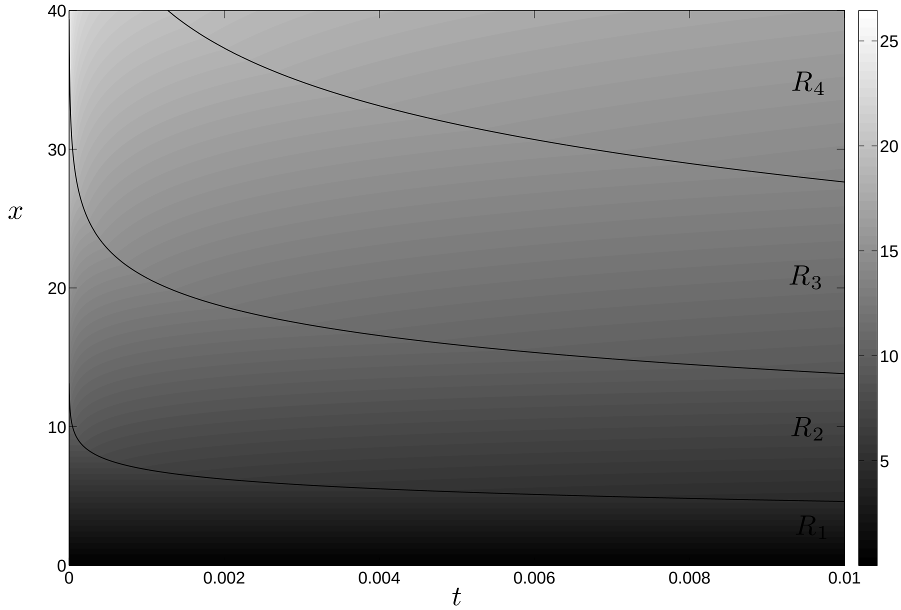

Figure 2: The small-t asymptotic approximation (15) of $K(x,t)$.

4 Discussion

An inductive method is used in the proof of Theorem 2 to find the limiting optimal fraction $K(x,t)/x$ of ammunition used as $t \to 0$. The same result holds when the bomber is restricted to only firing discrete units (integers, say) of ammunition x, the only modification of the proof needed is to replace x by $\lfloor x \rfloor$ (the largest integer $\leq x$) in the appropriate places.

Theorem 1 shows that $K(x,t) = x$ in a region asymptotically equivalent to R_1 in (16) and, this being monotone in x and R_1 being convex, this shows that conjecture [B] holds in this region. It is therefore natural to ask if the estimates of $K(x,t)$ in R_j given by Theorem 2 can be used to shed any light on conjecture [B] for $j \geq 2$. One thing we can say is that [B] is satisfied *in the limit* as $t \to 0$ in the following sense. Letting $x_1 \leq x_2$ be such that $\lim_{t\to 0} |\log t|/x_1 \in R_j$ and $\lim_{t\to 0} |\log t|/x_2 \in R_{j'}$ for some $j \leq j'$, by (15) we have

$$K(x_2,t) - K(x_1,t) \sim \frac{x_2}{j'} + \binom{j'-1}{2} |\log t| - \left[\frac{x_1}{j} + \binom{j-1}{2} |\log t| \right]. \tag{17}$$

If $j = j'$, then (17) is $(x_2 - x_1)/j \geq 0$. If $j < j'$, then (17) divided by $|\log t|$ is

$$\begin{aligned} \frac{x_2}{j' |\log t|} - \frac{x_1}{j |\log t|} + \frac{j'-j}{2} &> \left(\frac{\binom{j'}{2}}{j'} - \frac{\binom{j+1}{2}}{j} + \frac{j'-j}{2} \right) (1 + o(1)) \\ &= (j' - j - 1)(1 + o(1)) \end{aligned}$$

which approaches a nonnegative limit. However, to make this arguments hold for, say, all $x_1 \leq x_2$ sufficiently large and all t sufficiently small, higher order asymptotics are needed. In particular, the rate of convergence in (12) as a function of x and t is needed, for which the tools developed in the proofs of Theorems 1 and 2 may be a starting point.

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