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EFFICIENCY OF SEQUENTIAL HYPOTHESES TESTING

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Abstract. The classic work of Aivazjan (1959) gives an asymptotic comparison of the optimal properties of fixed-size and sequential tests for regular statistical experiments based on independent identically distributed observations. The aim of the present paper is to extend the results of Aivazjan to a much broader class of statistical experiments with local asymptotically normal (LAN) behaviour. In particular, they apply to the case of non-regular "almost smooth" families as well as to Markov dependent observations.

The classic work of Aivazjan [1] deals with an asymptotic comparison of the optimal fixed-size and sequential tests for regular statistical experiments based on independent identically distributed observations. The aim of the present paper is to extend the results of Aivazjan to a much broader class of statistical experiments with local asymptotically normal (LAN) behaviour. In particular, they apply to the case of non-regular LAN experiments (see [4], Ch. 2) and Markov dependent observations [8], [7].

Let $X_1, X_2, X_3, \ldots$ be a discrete-time stochastic process whose distribution is known up to a real-valued parameter θ. Let $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$ be two simple hypotheses about the parameter to be tested on the base of the observations $X_1, X_2, X_3, \ldots, X_\nu$, where ν is a stopping time. Let $\varphi = \varphi(X_1, X_2, \ldots, X_\nu)$ be a test of H_0 against H_1, i.e. a measurable function taking values in $[0, 1]$ (with the usual interpretation as the conditional probability, given observations $X_1, X_2, \ldots, X_\nu$, to reject H_0). The quantities $\alpha(\varphi) = E_{\theta_0}\varphi(X_1, X_2, \ldots, X_\nu)$ and $\beta(\varphi) = E_{\theta_1}(1 - \varphi(X_1, X_2, \ldots, X_\nu))$ are known as probabilities of the first and second kind, respectively.

One of the classic problems is to find an (optimal) test φ which would satisfy $\alpha(\varphi) \le \alpha$ and $\beta(\varphi) \le \beta$ for any given α and β.

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In the class of the fixed-size tests ($\nu \equiv n = \text{const}$), the solution is given by the well-known Neyman-Pearson test:

$$\varphi^* = \begin{cases} 1, & \text{if } Z_n > c, \\ \gamma, & \text{if } Z_n = c, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where

$$Z_n = Z_n(X_1, X_2, \ldots, X_n) = \frac{f^n(X_1, X_2, \ldots, X_n; \theta_1)}{f^n(X_1, X_2, \ldots, X_n; \theta_0)}$$

is the likelihood ratio statistics, $f^n(X_1, X_2, \ldots, X_n; \theta)$ being probability density function of $(X_1, X_2, \ldots, X_n)$ when θ is the true value of the parameter. Let $n^* = n^*(\alpha, \beta)$ be a minimal sample size n for which there exists a test φ such that $\alpha(\varphi) \le \alpha$ and $\beta(\varphi) \le \beta$. From the optimal property (see, for example, [3]) of the Neyman-Pearson test it follows that n^* can be determined as the minimal sample size n for which test (1) satisfies $\alpha(\varphi^*) \le \alpha$ and $\beta(\varphi^*) \le \beta$.

As well, when the observations $X_1, X_2, \ldots$ are independent and identically distributed the following is the classical result of Wald (see, for example, [3]). Let stopping time ν be defined as

$$\nu = \min\{n : Z_n \notin (a, b)\} \tag{2}$$

with some constants a, b, and let

$$\varphi = \varphi(X_1, X_2, \ldots, X_\nu) = \begin{cases} 1, & \text{if } Z_\nu \ge b, \\ 0, & \text{if } Z_\nu \le a. \end{cases} \tag{3}$$

If the constants a and b are chosen in such a way that $\alpha(\varphi) = \alpha$ and $\beta(\varphi) = \beta$ for test (3) then among all the tests whose error probabilities of the first and the second kind do not exceed α and β, respectively, test (3) has the minimal average sample number (ASN), both under hypothesis H_0 and under H_1.

In paper [1], the asymptotic behaviour of the two competitive test was investigated in the case of close hypotheses ($\theta_1 \to \theta_0$), under quite restrictive conditions of regularity and when the observations are independent and identically distributed. It was proved that if $\theta_1 = \theta_0 + \varepsilon$, $\varepsilon \to 0$, then

$$n^* \sim \frac{(\Phi^{-1}(1-\alpha) + \Phi^{-1}(1-\beta))^2}{\varepsilon^2 I(\theta_0)} \tag{4}$$

where Φ^{-1} is the inverse of the standard normal distribution function Φ, and $I(\theta)$ is the Fisher information.

As to the behaviour of the Wald test, in paper [1], it was shown that

$$E_{\theta_0}\nu \sim 2\frac{w(\alpha, \beta)}{\varepsilon^2 I(\theta_0)} \quad \text{and} \quad E_{\theta_1}\nu \sim 2\frac{w(\beta, \alpha)}{\varepsilon^2 I(\theta_0)}, \quad \varepsilon \to 0, \tag{5}$$

$w(x,y)$ being the Wald function

$$w(x,y) = x \ln \frac{x}{1-y} + (1-x) \ln \frac{1-x}{y}, \quad 0 < x < 1, \quad 0 < y < 1. \tag{6}$$

We extend these results to a broad class of locally asymptotically normal (LAN) statistical experiments assuming neither regularity of the experiment nor independence of the observations.

Let us assume that we are able to observe a discrete-time stochastic process $X_1, X_2, \ldots, X_n, \ldots$ with a distribution P_θ known up to a real-valued parameter θ, $\theta \in \Theta$, $\Theta \subset \mathbb{R}$. Let P_θ^n be the distribution of the first n observation $(X_1, X_2, \ldots, X_n)$, and let θ_0 be an interior point of Θ.

The LAN condition (cf.[5], [6], [4], [8]). *Let us say that the family $\{P_\theta\}$ is locally asymptotically normal (LAN) at the point θ_0 if there exists $n(\varepsilon)$ such that for any $t_\varepsilon \to t$, $\varepsilon \to 0$,*

$$Z_n^\varepsilon = \frac{dP_{\theta_0+\varepsilon}^n}{dP_{\theta_0}^n} = \exp\{\sqrt{t}\xi_n - t/2 + \psi_n\}, \tag{7}$$

with $n = [t_\varepsilon n(\varepsilon)]$, where ξ_n tends in distribution to a standard normal random variable ξ ($\xi_n \Rightarrow \xi \sim N(0,1)$), and $\psi_n \to 0$ in probability P_{θ_0}, when $X_1, X_2, \ldots$ follow P_{θ_0}.

Common examples of LAN families are: regular statistical experiments with independent observations (see [5], [6]), "almost smooth" statistical experiments ([4], Chapter 2), regular Markov dependent observations (see, for example, [8]), etc.

Properly saying, usually the LAN condition is formulated with $t_\varepsilon \equiv 1$ in (7) (see, for example, [4]). We will need this slightly modified variant, which normally holds under the usual LAN (as considered in [4]) because of independence of observations. In what follows we shall see, that the behaviour as in (7) is quite typical for a LAN experiment even without the independence condition.

In the case when $P_{\theta_0+\varepsilon}^n$ and $P_{\theta_0}^n$ are not absolutely continuous with respect to each other, we shall suppose that there is on the left-hand side of (7) the Radon-Nikodym derivative of the absolutely continuous part of $P_{\theta_0+\varepsilon}^n$ with respect to $P_{\theta_0}^n$. For simplicity, throughout the paper we shall suppose the absolute continuity, although all results will be valid without this condition.

Throughout the paper we will focus on the problem of testing the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_\varepsilon$ where $\theta_\varepsilon = \theta_0 + \varepsilon$, and $\varepsilon \to 0$.

Let $n^* = n^*(\varepsilon) = n^*(\varepsilon; \alpha, \beta)$ be a minimal integer n for which there exists a fixed-sample test of H_0 vs. H_1 based on n observations whose error probabilities of the first and the second kind do not exceed α and β, respectively. The following theorem is a quite easy generalization of (4).

Theorem 1. *Under condition* (7), *for* $n^* = n^*(\varepsilon)$ *the following equivalence is valid:*

$$n^*(\varepsilon) \sim n(\varepsilon)\left(\Phi^{-1}(1-\alpha)+\Phi^{-1}(1-\beta)\right)^2, \; \varepsilon \to 0\,. \tag{8}$$

Proof. First of all, we prove that

$$\varlimsup_{\varepsilon\to 0}\frac{n^*(\varepsilon)}{n(\varepsilon)} < \infty. \tag{9}$$

For that, is sufficient to find a test based on a number $[kn(\varepsilon)]$ of observations with error probabilities less than α and β, respectively. Let φ_k be a test based on the $n = [kn(\varepsilon)]$ observations rejecting H_0 when $Z_n^\varepsilon > 1$. Let us evaluate its error probabilities and their asymptotic behaviour. Because of (7) we have:

$$\alpha(\varphi_k) = P_{\theta_0}(Z_n^\varepsilon > 1) \to P(e^{\sqrt{k}\xi - k/2} > 1) = 1 - \Phi(\sqrt{k}/2),$$

and

$$\beta(\varphi_k) = P_{\theta_1}(Z_n^\varepsilon < 1) = E_{\theta_0} Z_n^\varepsilon I_{\{Z_n^\varepsilon < 1\}} \to Ee^{\sqrt{k}\xi - k/2} I_{\{e^{\sqrt{k}\xi - k/2} < 1\}} = 1 - \Phi(\sqrt{k}/2).$$

So, choosing k in such a way that $1 - \Phi(\sqrt{k}/2) < \min\{\alpha, \beta\}$, we have

$$\lim_{\varepsilon\to 0}\alpha(\varphi_k) = \lim_{\varepsilon\to 0}\beta(\varphi_k) < \min\{\alpha,\beta\},$$

and therefore $n^*(\varepsilon) < kn(\varepsilon)$ for ε small enough, which proves (9).

Suppose now that $\varepsilon_i \to 0$ is any sequence such that $n^*(\varepsilon_i)/n(\varepsilon_i) \to t$, as $i \to \infty$, for some $t < \infty$. Let us show that

$$t = (\Phi^{-1}(1-\alpha) + \Phi^{-1}(1-\beta))^2, \tag{10}$$

which would prove the theorem.

Let the constant $c = c(\varepsilon, n)$ be defined for each ε, n from the equality

$$P_{\theta_0}(Z_n^\varepsilon > c) + \gamma P_{\theta_0}(Z_n^\varepsilon = c) = \alpha. \tag{11}$$

with some $0 \le \gamma < 1$.

This definition is suggested by the form of the Neyman-Pearson test for H_0 vs. H_1 of level α.

We note first that for any $t_\varepsilon \to t$ as $\varepsilon \to 0$

$$c(\varepsilon, [t_\varepsilon n(\varepsilon)]) \to c,$$

where c is defined by the equality

$$P(e^{\sqrt{t}\xi - t/2} > c) = \alpha, \tag{12}$$

i.e.

$$c = \exp\{-t/2 + \sqrt{t}\Phi^{-1}(1-\alpha)\}. \tag{13}$$

This is quite obvious, because, as seen from (11), c is essentially the upper α-point of the distribution of the Z_n^ε statistics which converges weakly to that of the left-hand side in (12).

From the optimal property of the Neyman-Pearson test it follows that for $n = n^* = n^*(\varepsilon)$ it holds

$$P_{\theta_\varepsilon}(Z_n^\varepsilon < c(\varepsilon, n)) + (1-\gamma)P_{\theta_\varepsilon}(Z_n^\varepsilon = c(\varepsilon, n)) \quad \leq \quad \beta, \tag{14}$$

$$P_{\theta_\varepsilon}(Z_{n-1}^\varepsilon < c(\varepsilon, n-1)) + (1-\gamma)P_{\theta_\varepsilon}(Z_{n-1}^\varepsilon = c(\varepsilon, n-1)) \quad > \quad \beta\,. \tag{15}$$

Passing in (14) and (15) to the limit as $\varepsilon \to 0$, we get:

$$\lim_{\varepsilon\to 0} P_{\theta_\varepsilon}(Z_n^\varepsilon < c(\varepsilon, n)) = Ee^{\sqrt{t}\xi - t/2} I_{\{e^{\sqrt{t}\xi - t/2} < c\}} = \beta, \tag{16}$$

so that

$$\Phi\left(\ln \frac{c - t/2}{\sqrt{t}}\right) = \beta. \tag{17}$$

Combining (13) and (17) we get (10), which completes the proof. □

To treat the sequential case we need a condition of more complicated nature than (7). The following condition can be called the functional LAN condition.

Let $Z^\varepsilon(t)$ be a random function on $[0, \infty)$ defined as

$$Z^\varepsilon(t) = Z^\varepsilon_{[tn(\varepsilon)]}\,. \tag{18}$$

Let $D[0,T]$ be the Skorokhod space of right-continuous functions with left limits on $[0,T]$ endowed with the Skorokhod metric ρ.

The functional LAN condition. *For each $T > 0$ the distribution of the random function $Z^\varepsilon(t)$ in $D[0,T]$ converges weakly, when $X_1, X_2, \dots$ follow the distribution P_{θ_0}, to that of the random function $Z_0(t) = \exp\{w(t) - t/2\}$, where w is a standard Wiener process.*

The functional LAN condition is related to the LAN condition in the same way as the invariance principle of Donsker-Prokhorov (see, e.g., [2]) is related to the central limit theorem. As easily seen, representation (7) gives the weak convergence of one-dimensional distributions of Z^ε to those of Z_0.

In fact, the functional LAN condition is a simple consequence of the LAN condition if the observations $X_1, X_2, \dots,$ are independent. This is due to [9], see Theorem 1 therein. In particular, it holds for "almost smooth" distributions families considered in [4], Chapter 2. Moreover, it holds for regular families generated by Markov observations, considered, for example, in [8], due to a recent result by the author in [7].

In what follows we will show that the functional LAN condition implies that under the alternative hypothesis, i.e. when $X_1, X_2, \dots$ follow P_{θ_ε}, the limiting distribution of Z^ε is that of $Z_1(t) = \exp\{w(t) + t/2\}$.

With respect to Wald's SPRT under the functional LAN condition let us define, for any two constants (a,b), $a<1<b$, a stopping time

$$\nu^\varepsilon = \min\{n : Z_n^\varepsilon \notin (a,b)\}, \tag{19}$$

and let

$$\varphi^\varepsilon = \varphi^\varepsilon(X_1, X_2, \ldots, X^\varepsilon_{\nu^\varepsilon}) = \begin{cases} 1, & \text{if } Z^\varepsilon_{\nu^\varepsilon} \geq b, \\ 0, & \text{if } Z^\varepsilon_{\nu^\varepsilon} \leq a. \end{cases} \tag{20}$$

We will need also "limit" versions of the stopping time ν^ε in (19). Let

$$\nu_0 = \min\{n : Z_0(t) \notin (a,b)\}, \qquad \nu_1 = \min\{n : Z_1(t) \notin (a,b)\}, \tag{21}$$

In order that the error probabilities of test (19)-(20) be no greater than α and β, respectively, A.Wald proposed approximations

$$a = \frac{\beta}{1-\alpha}, \qquad b = \frac{1-\beta}{\alpha}, \tag{22}$$

(see e.g. [3]).

We will show that if the constants are chosen in such a way, the error probabilities of test (19)-(20) converge to α and β, respectively. Because, in view of Theorem 1, n^* is the minimal fixed sample size which provide the same error probabilities, α and β, respectively, it is of interest to investigate the asymptotic behaviour of the ASN of the latter test and compare it with n^*. The first is done in Theorem 2 below.

Let us denote $\mathfrak{L}(Z|P_\theta)$ the distribution of random element $Z = Z(X_1, X_2, \ldots)$ when $X_1, X_2, \ldots$ follow the distribution P_θ, or simply $\mathfrak{L}(Z)$ the distribution of Z when there is no ambiguity about the distribution.

Theorem 2. *Under the functional LAN condition, if the constants of test (19)-(20) are defined as in (22), then*

$$\alpha(\varphi^\varepsilon) \to \alpha, \qquad \beta(\varphi^\varepsilon) \to \beta, \quad \text{as} \quad \varepsilon \to 0\,. \tag{23}$$

As to ν^ε we have the following weak convergence of distributions:

$$\mathfrak{L}(\nu^\varepsilon|P_{\theta_0}) \to \mathfrak{L}(\nu_0), \quad \mathfrak{L}(\nu^\varepsilon|P_{\theta_\varepsilon}) \to \mathfrak{L}(\nu_1), \quad \text{as} \quad \varepsilon \to 0, \tag{24}$$

and when ν^ε is uniformly integrable with respect to P_{θ_0} (P_{θ_ε}) then

$$E_{\theta_0}\nu^\varepsilon \sim n(\varepsilon)E\nu_0, \quad (E_{\theta_\varepsilon}\nu^\varepsilon \sim n(\varepsilon)E\nu_1,) \quad \text{as} \quad \varepsilon \to 0, \tag{25}$$

where

$$E\nu_0 = 2w(\alpha,\beta), \quad E\nu_1 = 2w(\beta,\alpha), \tag{26}$$

(see (6) for the definition of $w(x,y)$).

Proof. First of all, let us note that under the functional LAN condition

$$\mathfrak{L}(Z^\varepsilon|P_{\theta_\varepsilon}) \to \mathfrak{L}(Z_1) \quad \text{as} \quad \varepsilon \to 0, \tag{27}$$

in $D[0,T]$. The convergence of the finite-dimentional distributions corresponding to (27) is straightforward because of the functional LAN condition.

It remains to show the relative compactness of $\mathfrak{L}(Z^\varepsilon|P_{\theta_\varepsilon})$. This follows from relative compactness of $\mathfrak{L}(Z^\varepsilon|P_{\theta_0})$ because of contiguity of $P^n_{\theta_0}$ and $P^n_{\theta_\varepsilon}$ for $n = [Tn(\varepsilon)]$ (see for the notion of contiguity e.g. [8]):

The contiguity is due to Proposition 3.1 of Chapter 1 in [8]. By virtue of the functional LAN convergence, for each $\delta > 0$ there exists a compact subset K_δ such that $P_{\theta_0}(Z^\varepsilon \notin K_\delta) < \delta$ for all ε (tightness). Let δ be any positive number. Then for any ε and K

$$\begin{aligned} P_{\theta_\varepsilon}(Z^\varepsilon \notin K) &= E_{\theta_0} Z^\varepsilon_n I_{\{Z^\varepsilon \notin K\}} \\ &= E_{\theta_0} Z^\varepsilon_n I_{\{Z^\varepsilon_n > c\}} I_{\{Z^\varepsilon \notin K\}} + E_{\theta_0} Z^\varepsilon_n I_{\{Z^\varepsilon_n \le c\}} I_{\{Z^\varepsilon \notin K\}} \\ &\le E_{\theta_0} Z^\varepsilon_n I_{\{X^\varepsilon_n > c\}} + c P_{\theta_0}\{Z^\varepsilon \notin K\}\,. \end{aligned} \tag{28}$$

Because of the contiguity Z^ε_n is uniformly integrable, so the first term on the right-hand side of (28) can be done less than $\delta/2$ by choosing c. Now, choosing a compact K in such a way that $P_{\theta_0}\{Z^\varepsilon \notin K\} < \delta/(2c)$ for all ε we will have that the right-hand side of (28) is less than δ for any ε. So, the tightness of the distribution of Z^ε is proved. Thus, by the theorem of Prokhorov ([2], Chapter 1, Theorem 6.1) $\mathfrak{L}(Z^\varepsilon|P_{\theta_\varepsilon})$ is relative compact, and hence converges weakly to $\mathfrak{L}(Z_1)$.

Let us prove now the first statement (23) of the Theorem. Let $A \subset D[0,T]$ be the subset of all functions x which "exit" the interval (a,b) through the upper bound. It is obvious that the boundary, in $D[0,T]$, of A is contained in

$$\{x \in D[0,T] : \sup_{t\in[0,T]} x(t) = b\}$$

which obviously has P_{Z_0}-measure 0 (as usual, P_Z stands for the distribution of Z). So from the theorem of Alexandroff [2] it follows that

$$\alpha(\varphi^\varepsilon) = P_{\theta_0}(Z^\varepsilon \in A) \to P(Z_0 \in A). \tag{29}$$

Analogously, $\beta(\varphi^\varepsilon) = P(Z^\varepsilon_n \in B)$, where B is the subset of $D[0,T]$ consisting of functions which exit the interval (a,b) through the lower bound, and its boundary has P_{Z_1}-measure 0, so

$$\beta(\varphi^\varepsilon) = P_{\theta_\varepsilon}(Z^\varepsilon \in B) \to P(Z_1 \in B). \tag{30}$$

As $Z_0(t)$ is a martingale with $EZ_0(t) = 1$, then $EZ_0(\nu_0) = 1$. Therefore

$$P(Z_0 \in A)b + (1 - P(Z_0 \in A))a = 1. \tag{31}$$

Analogously, Z_1^{-1} is a martingale, so

$$P(Z_1 \in B)\frac{1}{b} + (1 - P(Z_1 \in B))\frac{1}{a} = 1. \tag{32}$$

From (31) and (22) it follows immediately that $P(Z_0 \in A) = \alpha$, and analogously from (32) and (22) we get $P(Z_1 \in B) = \beta$. Thus, combining this with (29) and (30) we have

$$\lim_{\varepsilon \to 0} \alpha(\varphi^\varepsilon) \to \alpha, \qquad \lim_{\varepsilon \to 0} \beta(\varphi^\varepsilon) \to \beta, \quad \text{as} \quad \varepsilon \to 0,$$

which proves (23).

Relations (24) are immediate because the first exit time from the bound is an almost sure continuous functional on $D[0,T]$, and so (24) is a simple consequence of Theorem 5.1, Chapter 1, [2].

Now, relations (25) follow from (24) and from Theorem 5.4, Chapter 1, [2].

To prove (26) let us start with the identity

$$E \ln Z_0(\nu_0) = \alpha \ln b + (1 - \alpha) \ln a. \tag{33}$$

On the other hand,

$$E \ln Z_0(\nu_0) = E(w(\nu_0) - \nu_0/2) = -E\nu_0/2. \tag{34}$$

Combining (33) and (34) with (22) we get

$$E\nu_0 = 2\left(\alpha \ln \frac{\alpha}{1-\beta} + (1-\alpha) \ln \frac{1-\alpha}{\beta}\right)$$

which proves the first of relations (26). The other is proved in a similar way. □

Comparing the asymptotics of the ASN of the two competitive tests in Theorem 1 and Theorem 2 we have the following asymptotic efficiency of the Wald test with respect to the Neyman-Pearson test:

$$\lim_{\varepsilon \to 0} \frac{E_{\theta_0} \nu^\varepsilon}{n^*(\varepsilon)} = \frac{2w(\alpha, \beta)}{\left(\Phi^{-1}(1-\alpha) + \Phi^{-1}(1-\beta)\right)^2} \qquad \text{and} \tag{35}$$

$$\lim_{\varepsilon \to 0} \frac{E_{\theta_\varepsilon} \nu^\varepsilon}{n^*(\varepsilon)} = \frac{2w(\beta, \alpha)}{\left(\Phi^{-1}(1-\alpha) + \Phi^{-1}(1-\beta)\right)^2}. \tag{36}$$

This is a portion of the sample number which would require the Wald's SPRT, in average, in comparison with the non-sequential Neyman-Pearson test. The numeric evaluation of (35)-(36) can be found in [1].

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